MATH 7220 Homework 6

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1 Problem 11.1

Let A be a ring, M an A-module. The support of M is defined to be the set $\mathrm{Supp}(M)$ of prime ideals $\mathfrak{p}$ in A such that $M_\mathfrak{p} \neq 0$. Prove the following results:

1. $M \neq 0$ iff $\mathrm{Supp}(M) \neq \emptyset$.
2. $V(\mathfrak{a}) = \mathrm{Supp}(A/\mathfrak{a})$.
3. If $0 \to M' \to M \to M'' \to 0$ is exact, then $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$.
4. If $M = \sum M_i$, then $\mathrm{Supp}(M) = \bigcup \mathrm{Supp}(M_i)$.
5. If M is finitely generated, then $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$.
6. If M, N are finitely generated, then $\mathrm{Supp}(M \otimes_A N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$.
7. If M is finitely generated and $\mathfrak{a}$ is an ideal, then $\mathrm{Supp}(M/\mathfrak{a}M) = V(\mathfrak{a} + \mathrm{Ann}(M))$.
8. If $f : A \to B$ is a ring morphism and M is finitely generated, then $\mathrm{Supp}(B \otimes_A M) = f^{*-1}(\mathrm{Supp}(M))$.

Proof.

1. We know that $M = 0$ iff $M_\mathfrak{p} = 0$ for all primes $\mathfrak{p}$, i.e. $\mathrm{Supp}(M) = \emptyset$. Thus $M \neq 0$ iff $\mathrm{Supp}(M) \neq \emptyset$.
2. $\mathfrak{p} \in \mathrm{Supp}(A/\mathfrak{a})$ iff $(A/\mathfrak{a})_\mathfrak{p} = A_\mathfrak{p}/\mathfrak{a}_\mathfrak{p} \neq 0$ iff $\mathfrak{a}_\mathfrak{p} \neq A_\mathfrak{p}$ iff $\mathfrak{a}_\mathfrak{p}$ has no units iff $\mathfrak{a} \subset \mathfrak{p}$ iff $\mathfrak{p} \in V(\mathfrak{a})$.
3. We know localization is an exact functor, so $0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$ is exact for all primes $\mathfrak{p}$ in A. Exactness implies $M_\mathfrak{p} = 0$ iff $M'_\mathfrak{p} = M''_\mathfrak{p} = 0$. Taking complements we get $M_\mathfrak{p} \neq 0$ iff $M'_\mathfrak{p} \neq 0$ or $M''_\mathfrak{p} \neq 0$, as desired.
4. $M_\mathfrak{p} = \sum (M_i)_\mathfrak{p}$. $M_\mathfrak{p} = 0$ iff $(M_i)_\mathfrak{p} = 0$ for all i. Taking the complement we get $M_\mathfrak{p} \neq 0$ iff $(M_i)_\mathfrak{p} \neq 0$ for some i, as desired.

5. From part 2, $V(\text{Ann}(M)) = \text{Supp}(A/\text{Ann}(M))$. $(A/\text{Ann}(M))_{\mathfrak{p}} = A_{\mathfrak{p}}/\text{Ann}(M)_{\mathfrak{p}} = A_{\mathfrak{p}}/\text{Ann}(M_{\mathfrak{p}})$, where we have used that localization commutes with annihilators for finitely generated modules. Now, $M_{\mathfrak{p}} \neq 0$ iff $\text{Ann}(M_{\mathfrak{p}}) \neq A_{\mathfrak{p}}$ iff $A_{\mathfrak{p}}/\text{Ann}(M_{\mathfrak{p}}) \neq 0$, so $\text{Supp}(M) = V(\text{Ann}(M))$.

6. Localization commutes with tensor products, so if $M_{\mathfrak{p}} = 0$ or $N_{\mathfrak{p}} = 0$, then $(M \otimes_A N)_{\mathfrak{p}} = M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} = 0$. We now show a more general result, that is, if $M \otimes_R \otimes N = 0$ for R a local ring and finitely generated R modules M, N, then $M = 0$ or $R = 0$. This is relevant since $A_{\mathfrak{p}}$ is local.

   Let R be local with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. Then for an R-module M, let $M_k = k \otimes_R M = M/\mathfrak{m}M$. By Nakayama's lemma, $M_k = 0$ implies $M = 0$. Now suppose $M \otimes_R N = 0$. Tensoring with k gives $M_k \otimes_k N_k = 0$. M_k and N_k are vector spaces since k is a field, and they are finite dimensional since M, N are finitely generated as modules. The dimension of a tensor product is the product of the dimensions. Thus $M_k = 0$ or $N_k = 0$, since if both were non-zero we could construct a basis for their tensor product. Nakayama's lemma then implies $M = 0$ or $N = 0$.

   Returning to the situation at hand, with $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} = 0$; since $A_{\mathfrak{p}}$ is local, we can apply the previous lemma to get $M_{\mathfrak{p}} = 0$ or $N_{\mathfrak{p}} = 0$. Taking complements, we get the desired equality $\text{Supp}(M \otimes_A N) = \text{Supp}(M) \cap \text{Supp}(N)$.

7. We have $M/\mathfrak{a}M = M \otimes_A (A/\mathfrak{a})$, so by part 6 $\text{Supp}(M/\mathfrak{a}M) = \text{Supp}(M) \cap \text{Supp}(A/\mathfrak{a})$. By parts 2 and 5, $\text{Supp}(M) \cap \text{Supp}(A/\mathfrak{a}) = V(\text{Ann}(M)) \cap V(\mathfrak{a})$. Finally this is equal to $V(\text{Ann}(M) + \mathfrak{a})$ by properties of V.

8. Recall that $f^* : \text{Spec}(B) \to \text{Spec}(A)$ is given by $f^*(\mathfrak{q}) = f^{-1}(\mathfrak{q})$. It is a continuous map; in particular, $f^{*-1}(V(J)) = V(f(J)B)$. Thus $f^{*-1}(\text{Supp}(M)) = f^{*-1}(V(\text{Ann}(M))) = V(f(\text{Ann}(M))B)$. We first show that $V(\text{Ann}(B \otimes_A M)) \subset V(f(\text{Ann}(M))B)$.

   It suffices to show $f(\text{Ann}(M)) \subset \text{Ann}(B \otimes_A M)$. Indeed, if $y \in f(\text{Ann}(M))$, let $y = f(x)$ with $x \in \text{Ann}(M)$. Then $y(b \otimes m) = (f(a)b) \otimes m$. Recall that the natural A-module structure on B is given by $a \cdot b = f(a)b$. Thus $(f(a)b) \otimes m = (a \cdot b) \otimes m = b \otimes am = b \otimes 0 = 0$. Hence $y \in \text{Ann}(B \otimes_A M)$. Since $f(\text{Ann}(M)) \subset \text{Ann}(B \otimes_A M)$, $f(\text{Ann}(M))B \subset \text{Ann}(B \otimes_A M)$, and thus $V(\text{Ann}(B \otimes_A M)) \subset V(f(\text{Ann}(M))B)$. We have then shown $\text{Supp}(B \otimes_A M) \subset f^{*-1}(\text{Supp}(M))$.

   Not sure how to show the other direction.

$\square$

2 Problem 11.2

Let M be an A-module. Show that if M is finitely generated, then $\mathfrak{p} \in \mathrm{Supp}(M)$ iff $\mathfrak{p} \in V(\mathrm{Ann}(M))$.

Proof. This is exactly part 5 of the previous exercise. □

3 Problem 12.1 (#22)

Let A be a ring and $\mathfrak{p}$ a prime ideal of A. Then the canonical image of $\mathrm{Spec}(A_\mathfrak{p})$ in $\mathrm{Spec}(A)$ is equal to the intersection of all the open neighborhoods of $\mathfrak{p}$ in $\mathrm{Spec}(A)$.

Proof. Let $\phi : A \to A_\mathfrak{p}$ be the canonical morphism with induced map $\phi^* : \mathrm{Spec}(A_\mathfrak{p}) \to \mathrm{Spec}(A)$, so that the canonical image $X_\mathfrak{p} = \phi^*(\mathrm{Spec}(A_\mathfrak{p}))$. Points $x \in \mathrm{Spec}(A_\mathfrak{p})$ are prime ideals $\mathfrak{q}_x$ in $A_\mathfrak{p}$, which are in one-to-one correspondence (via ϕ) to prime ideals, which we will also call $\mathfrak{q}_x$, in A, such that $\mathfrak{q}_x \subset \mathfrak{p}$.

Now let U be some open neighborhood of $\mathfrak{p}$. The complement $X - U$ is equal to $V(\mathfrak{a})$ for some ideal $\mathfrak{a}$, and we have $\mathfrak{p} \notin V(\mathfrak{a})$. This means $\mathfrak{a} \not\subset \mathfrak{p}$. Since $\mathfrak{q}_x \subset \mathfrak{p}$, $\mathfrak{a} \subset \mathfrak{q}_x$ implies $\mathfrak{a} \subset \mathfrak{p}$. Thus $\mathfrak{a} \not\subset \mathfrak{q}_x$, so $\mathfrak{q}_x \notin V(\mathfrak{a})$, so $\mathfrak{q}_x \in U$. Thus $\mathfrak{q}_x$ is contained in all open neighborhoods of $\mathfrak{p}$.

Conversely, if a prime ideal $\mathfrak{q}$ in A is not contained in an open neighborhood U, then it is contained in the complement $V(\mathfrak{a})$, which means that $\mathfrak{q} \not\subset \mathfrak{p}$. Then $\mathfrak{q}$ does not correspond to some prime ideal in $A_\mathfrak{p}$. $\square$

4 Problem 12.2

Let A be a Noetherian ring, $\mathfrak{p}$ a prime ideal, and M a finitely generated A-module. Show that the following are equivalent:

1. $\mathfrak{p}$ belongs to 0 in M;
2. there exists nonzero $x \in M$ such that $\text{Ann}(x) = \mathfrak{p}$;
3. there exists a submodule of M isomorphic to $A/\mathfrak{p}$.

Deduce that there exists a chain of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ such that each quotient M_i/M_{i-1} is of the form $A/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a prime ideal of A.

Proof. ($2 \to 3$) The submodule (x) is isomorphic to $A/\mathfrak{p}$ via the map $f : A \to M$ given by $f(a) = ax$; the image is (x) and the kernel is $\mathfrak{p}$.

($3 \to 2$) Let $N \cong A/\mathfrak{p}$, $x \in N$, $x \neq 0$. Let $f : N \to A/\mathfrak{p}$ be an isomorphism. Let $a \in \mathfrak{p}$. Then $f(ax) = \overline{a}f(x) = 0$, where $\overline{a} = a \mod \mathfrak{p}$. Since f is an isomorphism, $ax = 0$. Thus $\mathfrak{p} \subset \text{Ann}(x)$. Conversely, if $ax = 0$, then $f(ax) = \overline{a}f(x) = 0$. Since $f(x) \neq 0$, $\overline{a} = 0$ since $A/\mathfrak{p}$ is an integral domain. Thus $a \in \mathfrak{p}$. Thus $\text{Ann}(x) = \mathfrak{p}$.

Not sure how to deal with part 1.

We know that such a prime $\mathfrak{p}$ exists, as it is an associated prime to 0 in M, and finitely generated modules over Noetherian rings have primary decompositions for their submodules. Thus let M_1 be a submodule of M isomorphic to $A/\mathfrak{p}_1$ for a prime $\mathfrak{p}_1$. In particular, we have $M_1/M_0 = A/\mathfrak{p}$ for $M_0 = 0$. M_1 is finitely generated, as we can identify it with (x) such that $\text{Ann}(x) = \mathfrak{p}$. Now pass to the quotient $M' = M/M_1$. M' is again a finitely generated A-module, so we have some submodule of M', $M_2' = A/\mathfrak{p}_2$. But this corresponds to some submodule M_2 of M such that $M_2/M_1 = M_2'$. Inductively, we can continue creating these submodules. This process must terminate since A is Noetherian. □

5 Problem 12.3

Find a minimal primary decomposition for the ideal (x^4, x^3y^4, x^3z^4) in $\mathbb{Q}[x, y, z]$. Which are the isolated and which are embedded primes? Do the same for $(y^5, y^2z^5, x^5y^2, x^3z^5, x^5z^4)$. In both cases, describe the zero sets of these ideals in projective 3-space.

Proof. We use Singular to create the decompositions. We get $(x^4, x^3y^4, x^3z^4) = (x^3) \cap (x^4, y^4, z^4)$, with the corresponding primes being (x) and (x, y, z). As $(x) \subset (x, y, z)$, (x) is isolated and (x, y, z) is embedded. Geometrically, the zero set corresponds to the plane $x = 0$, with a special embedded point at the origin.

Next we have $(y^5, y^2z^5, x^5y^2, x^3z^5, x^5z^4) = (y^2, z^4) \cap (y^2, x^3) \cap (x^5, y^5, z^5)$, with the corresponding primes being $(y, z), (x, y), (x, y, z)$. We have (y, z) and (x, y) contained in (x, y, z). Thus (y, z) and (x, y) are isolated and (x, y, z) is embedded. Geometrically, we must have $y = 0$, and then $x = 0$ or $z = 0$, so we have a union of two lines. The extra embedded component is the origin, which is the intersection of the two lines. □