MATH 7220 Homework 5

Andrea Bourque

February 2022

1 Problem 8.1

Let $F : \mathbf{A}^\circ \to \mathbf{Sets}$ be a contravariant functor. We say that F is representable if there is a pair (A, λ) such that $\lambda : h^A \to F$ is an isomorphism. Use Yoneda's lemma to show that if (B, μ) is another representing pair for F, there is a unique isomorphism $u : A \to B$ such that $\lambda = \mu \circ h^u$.

Proof. $\mu^{-1} \circ \lambda : h^A \to h^B$ uniquely corresponds to some $u : A \to B$ by Yoneda's lemma applied to h^B. In particular, $\mu^{-1} \circ \lambda = h^u$, so $\lambda = \mu \circ h^u$. Now we show that u is an isomorphism. Let $v : B \to A$ be the unique morphism such that $h^v = \lambda^{-1} \circ \mu$. By construction, $h^u \circ h^v$ and $h^v \circ h^u$ are identity natural transformations. The construction of the bijection in Yoneda's lemma takes the identity morphism to the identity natural transformation, meaning $h^u \circ h^v = h^{1_B}, h^v \circ h^u = h^{1_A}$. On the other hand, the Hom functors satisfy $h^u \circ h^v = h^{u \circ v}$, so by uniqueness, we have $u \circ v = 1_B, v \circ u = 1_A$, so that u is an isomorphism. $\square$

2 Problem 8.2

Let F be a field. Show that the ring $F[[x_1, ..., x_n]]$ of formal power series is a local ring with maximal ideal $(x_1, ..., x_n)$.

Proof. We use the following result: a ring is local iff the set of non-invertible elements is an ideal; this ideal is then the unique maximal ideal. An element of $F[[x_1, ..., x_n]]$ has an inverse iff it has a non-zero constant term. Then the ideal $(x_1, ..., x_n)$ consists exactly of the non-invertible elements, so we are done. $\square$

3 Problem 9.1

Give an example of two A-modules M and N such that $M_\mathfrak{p}$ and $N_\mathfrak{p}$ are isomorphic for all primes $\mathfrak{p}$, but M is not isomorphic to N.

Proof. Let $A = \mathbb{Z}[\sqrt{-5}]$. Let M be the ideal $(2, 1+\sqrt{-5})$, considered as an A module. We use the following: finitely generated modules over Dedekind domains are projective if and only if they are torsion free. Since A is a Dedekind domain and M is torsion free, M is projective. M itself is not free, since it is properly contained in A. Localizations of projective modules are projective over the localized ring. Furthermore, projective modules over local rings are free. Thus $M_\mathfrak{p}$ is always free, and isomorphic to $N_\mathfrak{p}$, where $N = A^n$ for some n. Since M is not free, M is not isomorphic to N. □

4 Problem 9.2

In the polynomial ring $K[x, y, z]$ where K is a field, let $\mathfrak{p}_1 = (x, y), \mathfrak{p}_2 = (x, z), \mathfrak{m} = (x, y, z)$. Let $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2$. Show that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a reduced primary decomposition of $\mathfrak{a}$. Which components are isolated and which are embedded?

Proof. $\mathfrak{a} = (x, y)(x, z) = (x^2, xy, xz, yz)$. $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (x, yz)$. $\mathfrak{m}^2 = (x^2, y^2, z^2, xy, xz, yz)$.

Then $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2 = (x^2, xy, xz, yz) = \mathfrak{a}$. Since $\mathfrak{p}_1, \mathfrak{p}_2$ are prime (they give quotients of $K[z], K[y]$ respectively), they are primary. Furthermore, powers of maximal ideals are primary, so $\mathfrak{m}^2$ is primary as well. Thus we have a primary decomposition. To show it is reduced, we note that the radicals are $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{m}$, which are all distinct. $\mathfrak{p}_1 \cap \mathfrak{p}_1 = (x, yz) \not\subset \mathfrak{m}^2$ since $x \notin \mathfrak{m}^2$. $\mathfrak{p}_1 \cap \mathfrak{m}^2 = (x^2, y^2, xy, xz, yz) \not\subset \mathfrak{p}_2$ since $y^2 \notin \mathfrak{p}_2$. Finally $\mathfrak{p}_2 \cap \mathfrak{m}^2 = (x^2, z^2, xy, xz, yz) \not\subset \mathfrak{p}_1$ since $z^2 \notin \mathfrak{p}_1$. Thus the primary decomposition is reduced.

The associated primes are $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{m}$. Clearly $\mathfrak{p}_1, \mathfrak{p}_2 \subset \mathfrak{m}$, whereas $\mathfrak{p}_1 \not\subset \mathfrak{p}_2$ and $\mathfrak{p}_2 \not\subset \mathfrak{p}_1$. Thus $\mathfrak{p}_1, \mathfrak{p}_2$ are isolated while $\mathfrak{m}$ is embedded. $\square$

5 Problem 10.1

Let A be a Noetherian ring and let M be a finitely generated A module. Let $\mathcal{P}$ be the set of annihilators of nonzero elements of M. Show that the maximal elements of $\mathcal{P}$ are prime.

Proof. Let $\mathfrak{p} \in \mathcal{P}$ be maximal; a maximal element exists because A is Noetherian. Let $\mathfrak{p} = \operatorname{Ann}(x)$. Suppose $yz \in \operatorname{Ann}(x)$ with $z \notin \operatorname{Ann}(x)$. Then $xz \neq 0$, so $\operatorname{Ann}(xz) \in \mathcal{P}$. Since $wx = 0$ implies $wxz = 0$, we have $\operatorname{Ann}(x) \subset \operatorname{Ann}(xz)$. Since $\operatorname{Ann}(x)$ is maximal, we have $\operatorname{Ann}(xz) = \operatorname{Ann}(x)$. $xyz = 0$, so $y \in \operatorname{Ann}(xz)$, so $y \in \operatorname{Ann}(x)$. Thus $\operatorname{Ann}(x)$ is prime. $\square$

6 Problem 10.2

Show that the radicals of the primary ideals in a minimal primary decomposition of $\mathfrak{a}$ are exactly the primes which are annihilators of nonzero $x \in A/\mathfrak{a}$.

Proof. In class, we showed that the primes of a minimal decomposition are exactly the primes of the form $(\mathfrak{a} : x)$ for $x \in A$, assuming A is Noetherian. But $y \in (\mathfrak{a} : x)$ iff $xy \in \mathfrak{a}$ iff $xy \equiv 0 \mod \mathfrak{a}$ iff $y \in \text{Ann}(x + \mathfrak{a})$. Since $x \equiv 0 \mod \mathfrak{a}$ implies $(\mathfrak{a} : x) = A$, which is not prime, we are done. $\square$