MATH 7220 Homework 4

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1 Problem 6.2

Now let $F : \mathbf{A}^\circ \to \mathbf{Set}$ be any contravariant functor. Prove Yoneda's Lemma: For any object A there is a canonical bijection $\alpha : \mathrm{Hom}_{\mathrm{Functors}}(h^A, F) \to F(A)$.

Proof. For $f : h^A \to F$, let $\alpha(f) = f(A)(1_A)$. We let $\beta : F(A) \to \mathrm{Hom}_{\mathrm{Functors}}(h^A, F)$ be defined as follows. For $x \in F(A)$, let $\beta(x)$ be such that $\beta(x)(B)$ sends $u \in h^A(B)$ to $F(u)(x)$. Now let $g : B \to C$ in $\mathbf{A}$. We must show $\beta(x)(B) \circ h^A(g) = F(g) \circ \beta(x)(C)$. Let $u \in h^A(C)$. Then $\beta(x)(B) \circ h^A(g)(u) = \beta(x)(B)(u \circ g) = F(u \circ g)(x)$, and $F(g) \circ \beta(x)(C)(u) = F(g)(F(u)(x)) = F(u \circ g)(x)$, where the last step follows from F being a contravariant functor. Thus $\beta(x)$ is a natural transformation.

$\alpha(\beta(x)) = \beta(x)(A)(1_A) = F(1_A)(x) = 1_{F(A)}(x) = x$, using that F is a functor in the second to last equality. For $f : h^A \to F$, $\beta(\alpha(f)) = \beta(f(A)(1_A))$ is a natural transformation. For objects B, $\beta(f(A)(1_A))(B)$ sends $u \in h^A(B)$ to $F(u)(f(A)(1_A))$. Using naturality of f, $F(u)(f(A)(1_A)) = f(B)(h^A(u)(1_A)) = f(B)(1_A \circ u) = f(B)(u)$. Thus $\beta(f(A)(1_A)) = f$ and we are done. $\square$

2 Problem 6.3

Show that for objects A, B in $\mathbf{A}$, there is a canonical bijection $\mathrm{Hom}_{\mathrm{Functors}}(h^A, h^B) \to \mathrm{Hom}_{\mathbf{A}}(A, B)$.

Proof. This is just Yoneda's lemma for $F = h^B$; $h^B(A) = \mathrm{Hom}(A, B)$. $\square$

3 Problem 6.4

Given a morphism $u : A \to B$ in $\mathbf{A}$, show that there are natural transformations $h^u : h^A \to h^B$ and $h_u : h_B \to h_A$.

Proof. Using the above canonical bijection $\mathrm{Hom}_{\mathrm{Functors}}(h^A, h^B) \to \mathrm{Hom}_{\mathbf{A}}(A, B)$, any $u : A \to B$ determines some natural transformation $h^u : h^A \to h^B$. Using the covariant form of Yoneda's lemma for the functor h_A, we have a natural bijection $\mathrm{Hom}_{\mathrm{Functors}}(h_B, h_A) \to h_A(B) = \mathrm{Hom}(A, B)$. Thus $u : A \to B$ determines a natural transformation $h_u : h_B \to h_A$. $\square$

4 Problem 7.1

Give examples of the following:

- An exact sequence of A-modules $0 \to M \to N \to P \to 0$ and an A-module T such that:

  1. $\mathrm{Hom}(T, N) \to \mathrm{Hom}(T, P)$ is not surjective.
  2. $\mathrm{Hom}(N, T) \to \mathrm{Hom}(M, T)$ is not surjective.
  3. $M \otimes T \to N \otimes T$ is not injective.

Proof. Throughout we take the exact sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \to 0$ of $\mathbb{Z}$ modules, and $T = \mathbb{Z}/2\mathbb{Z}$.

1. This gives $0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0$; $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$ because if $f(1) = n$ in $\mathbb{Z}$, we would have $2n = 2f(1) = f(0) = 0$. The 0 map into $\mathbb{Z}/2\mathbb{Z}$ is not surjective.

2. This gives $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \to 0$; $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ since there are two morphisms, one sending $1 \in \mathbb{Z}$ to $1 \in \mathbb{Z}/2\mathbb{Z}$, and the other sending 1 to 0. The map $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is given by $f \mapsto g$ such that $g(n) = f(2n)$. But $f(2n) = 0$ for $f \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$, so this is indeed the zero map. The 0 map into $\mathbb{Z}/2\mathbb{Z}$ is not surjective.

3. This gives $0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$; the multiplication by 2 map is 0 on $\mathbb{Z}/2\mathbb{Z}$. The zero map out of $\mathbb{Z}/2\mathbb{Z}$ is not injective. $\square$

5 Problem 7.2

Calculate $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ and $\mathrm{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ for all $i \geq 0$.

Proof. First we compute the $\mathrm{Ext}^i(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ and $\mathrm{Tor}_i(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ for $i \geq 0$. At $i = 0$ we have $\mathrm{Ext}^0 = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} = \mathrm{Tor}_0$. We take the exact sequence $0 \to \mathbb{Z} \xrightarrow{=} \mathbb{Z} \to 0 \to 0$.

For Ext we get

$$0 \to \mathrm{Hom}(0, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to ...$$

Replacing the Hom terms by what we know gives

$$0 \to 0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{=} \mathbb{Z}/n\mathbb{Z} \to \mathrm{Ext}^1(0, \mathbb{Z}/n\mathbb{Z}) \to ...$$

Since $0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{=} \mathbb{Z}/n\mathbb{Z} \to 0$ is exact, we see that this sequence must terminate and we get that $\mathrm{Ext}^i(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = 0$ for $i > 0$.

For Tor we get

$$0 \leftarrow 0 \otimes \mathbb{Z}/n\mathbb{Z} \leftarrow \mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}...$$

Replacing the tensor product terms by what we know gives

$$0 \leftarrow 0 \leftarrow \mathbb{Z}/n\mathbb{Z} \xleftarrow{=} \mathbb{Z}/n\mathbb{Z} \leftarrow \mathrm{Tor}_1(0, \mathbb{Z}/n\mathbb{Z}) \leftarrow ...$$

Since $0 \leftarrow \mathbb{Z}/n\mathbb{Z} \xleftarrow{=} \mathbb{Z}/n\mathbb{Z} \leftarrow 0$ is exact, we have that the sequence terminates and we get that $\mathrm{Tor}_i(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = 0$ for $i > 0$.

We now return to the general problem. Throughout, let $d = \gcd(m, n)$. For $i = 0$ we have $\mathrm{Ext}^0(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z} = \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} = \mathrm{Tor}_0(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$. In particular if $d = 1$, these groups are trivial. Take the exact sequence $0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/m\mathbb{Z} \to 0$.

For Ext we get

$$0 \to \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{\times m} \mathbb{Z}/n\mathbb{Z} \to \mathrm{Ext}^1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to 0 \to 0 \to \mathrm{Ext}^2 \to ...$$

since $\mathrm{Ext}^i(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = 0$ for $i > 0$. This implies $\mathrm{Ext}^i(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = 0$ for $i > 1$. Then by exactness, $\mathrm{Ext}^1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})/m(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$.

For Tor we get

$$0 \leftarrow \mathbb{Z}/d\mathbb{Z} \leftarrow \mathbb{Z}/n\mathbb{Z} \xleftarrow{\times m} \mathbb{Z}/n\mathbb{Z} \leftarrow \mathrm{Tor}_1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \leftarrow 0 \leftarrow 0 \leftarrow \mathrm{Tor}_2 \leftarrow ...$$

since $\mathrm{Tor}_i(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = 0$ for $i > 0$. This implies $\mathrm{Tor}_i(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = 0$ for $i > 1$. By exactness, $\mathrm{Tor}_1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ is the kernel of the multiplication by m map on $\mathbb{Z}/n\mathbb{Z}$, which is $\mathbb{Z}/d\mathbb{Z}$.

For readability (note $d = \gcd(m, n)$):

$$\mathrm{Ext}^i(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z} \text{ for } i = 0, 1; \mathrm{Ext}^i = 0 \text{ otherwise},$$

$$\mathrm{Tor}_i(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z} \text{ for } i = 0, 1; \mathrm{Tor}_i = 0 \text{ otherwise}.$$

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