# MATH 7220 Homework 3

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## 1 Problem 5.1

Let $\varphi : A \to B$ be a ring homomorphism and let $f : Y \to X$ be the induced map between $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. Recall that for $a \in A$, the set $D(a)$ consists of all prime ideals of $A$ which do not contain $a$. Show $f^{-1}(D(a)) = D(\varphi(a))$.

*Proof.* $Q \in f^{-1}(D(a))$ iff $f(Q) = \varphi^{-1}(Q) \in D(a)$ iff $a \notin \varphi^{-1}(Q)$ iff $\varphi(a) \notin Q$ iff $Q \in D(\varphi(a))$. □

## 2 Problem 5.2

Show that there is a canonical homomorphism $\theta_a : A[\frac{1}{a}] \to B[\frac{1}{\varphi(a)}]$.

*Proof.* Let $\lambda_A : A \to A[\frac{1}{a}], \lambda_B : B \to B[\frac{1}{\varphi(a)}]$ be the localization maps. Then $\lambda_B \circ \varphi$ is a homomorphism $A \to B[\frac{1}{\varphi(a)}]$. In particular, $a$ is mapped to $\varphi(a)/1$, which is invertible in $B[\frac{1}{\varphi(a)}]$. By the universal property of localization, $\lambda_B \circ \varphi$ must uniquely factor through $\lambda_A$, giving a unique homomorphism $\theta_a : A[\frac{1}{a}] \to B[\frac{1}{\varphi(a)}]$ such that $\lambda_B \circ \varphi = \theta_a \circ \lambda_A$. □

## 3 Problem 6.1

Let **A** be a category. For each object $A$ in **A** there are two functors to the category of sets: $h^A : \mathbf{A}^\circ \to \mathbf{Set}, h_A : \mathbf{A} \to \mathbf{Set}$. They are defined on objects by $h^A(B) = \mathrm{Hom}_{\mathbf{A}}(B, A), h_A(B) = \mathrm{Hom}_{\mathbf{A}}(A, B)$ Define the functors on morphisms.

*Proof.* Let $u : B \to C$. Then $h^A(u) : h^A(C) \to h^A(B)$ is given as follows. Let $g \in h^A(C)$, so that $g : C \to A$. Then $g \circ u : B \to A$ is in $h^A(B)$. Thus $h^A(u)(g) = g \circ u$.

$h_A(u) : h_A(B) \to h_A(C)$ is defined as follows. For $f \in h_A(B)$, $u \circ f : A \to C$ is in $h_A(C)$, so $h_A(u)(f) = u \circ f$. □