MATH 7220 Homework

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1 Problem 1

Show that for a subset $X \subset \mathbb{A}^n(k)$, $Z(I(X))$ is the Zariski-closure $\overline{X}$.

Proof. $Z(I(X))$ is an algebraic set by definition, so it is closed in the Zariski topology, also by definition. We have the inclusion $X \subset Z(I(X))$, because if $a \in X$, then for any $f \in I(X)$, $f(a) = 0$, so $a \in Z(I(X))$. Thus $\overline{X} \subset Z(I(X))$.

Now, as $\overline{X}$ is closed, we have $\overline{X} = Z(J)$ for some ideal J of the polynomial ring. Since $X \subset \overline{X} = Z(J)$, we know that for any $a \in X$ and any $f \in J$, $f(a) = 0$. Thus any $f \in J$ is also in $I(X)$. The inclusion $J \subset I(X)$ implies $Z(I(X)) \subset Z(J) = \overline{X}$, so we are done. □

2 Problem 2

Describe $\text{Spec}(\mathbb{R}[x])$. What about $\text{Spec}(k[x])$ for any field k?

Proof. First, we know that $k[x]$ is an integral domain. Hence $(0) \in \text{Spec}(k[x])$. Furthermore, $k[x]$ is a PID. Any nonzero ideal is of the form $(f(x))$ for some unique monic polynomial $f(x)$, and the ideal is prime iff $f(x)$ is irreducible over k. This is as far as we can go for arbitrary k.

For $k = \mathbb{R}$, the irreducible polynomials are $x - a$ and $x^2 - 2ax + a^2 + b^2$ for $a, b \in \mathbb{R}$. Indeed, whenever a polynomial in $\mathbb{R}[x]$ has non-real roots, they must come in conjugate pairs, giving the polynomial factors of $(x - (a + bi))(x - (a - bi)) = x^2 - 2ax + a^2 + b^2$. If the polynomial has a real root, then it has a factor of $x - a$. Any non-constant polynomial in $\mathbb{R}[x]$ has at least one (complex) root, so we are done. □

3 Problem 3

If $\varphi : A \to B$ is a ring homomorphism, show that the map $f = \varphi^a : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is continuous for the Zariski topology.

Proof. Let $X = \operatorname{Spec}(A), Y = \operatorname{Spec}(B)$. We show that the inverse image of a closed set in X is closed in Y. The closed sets in X are of the form $V(J)$ for ideals $J \subset A$.

We first show that for $Q \in Y$, an ideal $J \subset A$ satisfies $J \subset \varphi^{-1}(Q)$ iff $\varphi(J)B \subset Q$.

($\to$) Let $J \subset \varphi^{-1}(Q)$. Certainly then $\varphi(J) \subset Q$. Since Q is an ideal containing $\varphi(J)$, it must contain the ideal $\varphi(J)B$ which $\varphi(J)$ generates in B.

($\leftarrow$) Let $\varphi(J)B \subset Q$. $\varphi(J) \subset \varphi(J)B \subset Q$, so if $j \in J$ then $\varphi(j) \in Q$, so $J \subset \varphi^{-1}(Q)$.

Now we show that $(\varphi^a)^{-1}(V(J)) = V(\varphi(J)B)$: $Q \in (\varphi^a)^{-1}(V(J))$ iff $\varphi^a(Q) = \varphi^{-1}(Q) \in V(J)$ iff $J \subset \varphi^{-1}(Q)$ iff $\varphi(J)B \subset Q$ iff $Q \in V(\varphi(J)B)$. Thus the inverse image of closed sets are closed as desired. $\square$

4 Problem 4

Verify that $(a, s) \sim (b, t)$ (as defined in class) is an equivalence relation on $A \times S$. Then show that the formulas for addition and multiplication are well-defined.

Proof. Reflexive: $(a, s) \sim (a, s)$ because $as - as = 0$, so certainly there is some $u \in S$ such that $u(as - as) = 0$.

Symmetric: Suppose $(a, s) \sim (b, t)$ so that there is $u \in S$ such that $u(at - bs) = 0$. Then $u(bs - at) = 0$, which means $(b, t) \sim (a, s)$.

Transitive: Suppose $(a, s) \sim (b, t)$ so that there is $u \in S$ such that $u(at - bs) = 0$, and suppose $(b, t) \sim (c, v)$ so that there is $w \in S$ such that $w(bv - ct) = 0$. We cancel out the b terms by taking $0 = wvu(at - bs) + suw(bv - ct) = wut(av - cs)$. Since $w, u, t \in S$ and S is multiplicative, $wut \in S$, so this shows $(a, s) \sim (c, v)$.

Addition: By symmetry, it suffices to show that $(a, s) + (b, t) = (at + bs, st) \sim (av + cs, sv) = (a, s) + (c, v)$ when $(b, t) \sim (c, v)$. Thus let $u \in S$ be such that $u(bv - ct) = 0$. Then $u((at + bs)(sv) - (av + cs)(st)) = s^2 u(bv - ct) = 0$, so $(at + bs, st) \sim (av + cs, sv)$ as desired.

Multiplication: It again suffices to show that $(a, s)(b, t) = (ab, st) \sim (ac, sv) = (a, s)(c, v)$ when $(b, t) \sim (c, v)$. Thus let $u \in S$ be such that $u(bv - ct) = 0$. Then $u((ab)(sv) - (ac)(st)) = asu(bv - ct) = 0$, so $(ab, st) \sim (ac, sv)$. $\square$

5 Problem 5

Proof. □

6 Problem 6

Proof. □