MATH 7220 Homework 1

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1 Problem 1

Let $\varphi : \mathbb{Z}[x,y] \to \mathbb{Z}[t]$ be the homomorphism $\varphi(f(x,y)) = f(t^2,t^3)$. Show that $\ker\varphi = (y^2 - x^3)$.

Proof. We have $y^2 - x^3 \in \ker\varphi$ since $\varphi(y^2 - x^3) = (t^3)^2 - (t^2)^3 = t^6 - t^6 = 0$. Thus $(y^2 - x^3) \subset \ker\varphi$.

Let $f \in \ker\varphi$. We work mod $y^2 - x^3$, so that $f(x,y) \equiv g(x)y + h(x)$ for some $g, h \in \mathbb{Z}[x]$. Then $g(t^2)t^3 + h(t^2) = 0$. However, all the terms of $g(t^2)t^3$ have odd degree, and all the terms of $h(t^2)$ have even degree, implying that $g = h = 0$, and $f(x,y) \equiv 0 \bmod y^2 - x^3$. Thus $\ker\varphi = (y^2 - x^3)$. □

2 Problem 2

Let $\psi : \mathbb{Z}[x, y, z] \to \mathbb{Z}[t]$ be the homomorphism $\psi(f(x, y, z)) = f(t^3, t^4, t^5)$. Show that $\ker\psi = (xz - y^2, x^2y - z^2, x^3 - yz)$.

Proof. We have the inclusion $(xz - y^2, x^2y - z^2, x^3 - yz) \subset \ker\psi$ since $t^3t^5 - (t^4)^2 = (t^3)^2t^4 - (t^5)^2 = (t^3)^3 - t^4t^5 = 0$.

Let $f \in \ker\psi$. First, we work mod $x^2y - z^2$, so that $f(x, y, z) = g(x, y)z + h(x, y)$ for some $g, h \in \mathbb{Z}[x, y]$. Now let $g(x, y) = a + xp(x, y) + yq(y)$. Reducing mod $xz - y^2$ and $x^3 - yz$, we get $f = az + y^2p(x, y) + x^3q(y) + h(x, y)$. Applying ψ we have $at^5 + t^8p(t^3, t^4) + t^9q(t^4) + h(t^3, t^4) = 0$. Notice that $t^8p(t^3, t^4) + t^9q(t^4) + h(t^3, t^4)$ contains no terms of degree five, implying that $a = 0$. Thus $f = y^2p(x, y) + x^3q(y) + h(x, y) \in \mathbb{Z}[x, y]$. Since $x^4 - y^3 = x(x^3 - yz) + y(xz - y^2)$, we have $x^4 = y^3$ in the ideal, so $f = b(x)y^2 + c(x)y + d(x)$ and thus $b(t^3)t^8 + c(t^3)t^4 + d(t^3) = 0$. The terms of $b(t^3)t^8$ have degrees which are 2 mod 3. The terms of $c(t^3)t^4$ have degrees which are 1 mod 3. The terms of $d(t^3)$ have degrees which are 0 mod 3. Thus we must have $b(x) = c(x) = d(x) = 0$, so that $f \in (xz - y^2, x^2y - z^2, x^3 - yz)$. Finally, we have shown $\ker\psi = (xz - y^2, x^2y - z^2, x^3 - yz)$. □

3 Problem 3

What does $\{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 = 1\} \subset \mathbb{A}^2(\mathbb{C})$ look like?

Proof. Let $x^2 + y^2 = 1$. We can factor $(x + iy)(x - iy) = 1$. Introducing the substitution $u = x + iy, v = x - iy$ we have $uv = 1$, $x = \frac{u+v}{2}, y = \frac{u-v}{2i}$. Thus $\{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 = 1\} \cong \{(u, v) \in \mathbb{C}^2 \mid uv = 1\}$. Next, given $u \in \mathbb{C} - \{0\}$, we can always find $v = 1/u$ so that $uv = 1$. Similarly, if $uv = 1$, then $u \in \mathbb{C} - \{0\}$. Thus $\{(u, v) \in \mathbb{C}^2 \mid uv = 1\} \cong \mathbb{C} - \{0\}$. Since $\mathbb{C}$ can be considered a once-punctured sphere via stereographic projection, $\mathbb{C} - \{0\}$ is then a twice-punctured sphere. $\square$

4 Problem 4

Prove that the following conditions on an R-module M, where R is a commutative ring, are equivalent:

1) Every submodule of M is finitely generated.

2) Every ascending chain of submodules of M terminates.

3) Every set of submodules of M contains a maximal element under inclusion.

4) Given any sequence of elements $f_1, f_2, ... \in M$, there is a number m such that for each $n > m$ there is an expression $f_n = \sum_{i=1}^{m} a_i f_i$ with $a_i \in R$.

Proof. ($1 \to 2$) Let $N_1 \subset N_2 \subset ...$ be an ascending chain of submodules. We first show that $N = \bigcup_{i=1}^{\infty} N_i$ is a submodule of M. For two elements $x, y \in N$, we have $x \in N_i, y \in N_j$ for some $i, j \in \mathbb{N}$. WLOG, suppose $i \leq j$, so that $x \in N_j$. Then all the module conditions are satisfied since x, y are elements of the same submodule of M. By hypothesis N is finitely generated, say $N = (x_1, ..., x_n)$. Each x_i is contained in some N_{j_i} for $i = 1, ..., n$. Taking j to be the maximum of $j_1, ..., j_n$, then $x_i \in N_j$ for each $i = 1, .., n$. Then $N = (x_1, ..., x_n) \subset N_j \subset N_{j+1} \subset ... \subset N$, implying $N_j = N_{j+1} = ... = N$.

($2 \to 3$) Let $\mathcal{N}$ be a collection of submodules of M. Let $N_1 \in \mathcal{N}$. If there is no $N \in \mathcal{N}$ containing but not equal to N_1, then N_1 is maximal and we are done. Otherwise, let $N_2 \in \mathcal{N}$ such that $N_1 \subset N_2$. We repeat this process. If at some point we have $N_1 \subset N_2 \subset ...N_i$ and N_i is maximal, we are done. Otherwise, suppose we repeat this process indefinitely. We get an ascending chain $N_1 \subset N_2 \subset ...$ of submodules. By hypothesis, this chain must terminate. In particular, $N_i = N_{i+1} = ...$ for some $i \in \mathbb{N}$. This contradicts the choice of N_{i+1} to be not equal to N_i, implying N_i is maximal.

($3 \to 4$) Let $f_1, f_2, ... \in M$. Consider the collection of submodules $\{(f_1), (f_1, f_2), ...\}$. By hypothesis there is a maximal element, which is of the form $(f_1, f_2, ..., f_m)$ for some $m \in \mathbb{N}$. However, if $n > m$, we trivially have that $(f_1, f_2, ..., f_m) \subset (f_1, ..., f_n)$. By the maximal condition, this means $(f_1, f_2, ..., f_m) = (f_1, ..., f_n)$. Thus $f_n \in (f_1, ..., f_m)$, implying there is an expression $f_n = \sum_{i=1}^{m} a_i f_i$ with $a_i \in R$.

($4 \to 1$) Let N be a submodule of M. Let $f_1 \in N$. If $(f_1) = N$, we are done. If $N = (f_1, ..., f_i)$ for $i \geq 1$, we are done. Otherwise, we can choose $f_{i+1} \in N - (f_1, ..., f_i)$ for all i. In such a way we have a sequence of elements $f_1, f_2, ... \in N \subset M$. By hypothesis, there is $m \in \mathbb{N}$ such that for $n > m$, $f_n = \sum_{i=1}^{m} a_i f_i$ for $a_i \in R$. But this means that $f_{m+1} \in (f_1, ..., f_m)$, a contradiction. Thus N must be finitely generated. □