MATH 7211 Homework 9

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May 9, 2023

1 Problem 1

Let $\mathbb{Z}_n$ denote the ring $\mathbb{Z}/n\mathbb{Z}$ for any positive integer n. Prove that $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z}_m \cong \mathbb{Z}_d$, where $d = \gcd(m, n)$.

Proof. Consider the map $\mathbb{Z}_n \times \mathbb{Z}_m \to \mathbb{Z}_d$, which is defined by $(a \mod n, b \mod m) \mapsto ab \mod d$. This is well-defined because $d \mid n$ and $d \mid m$, and it is clearly $\mathbb{Z}$-bilinear. Thus it induces a unique map $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z}_m \to \mathbb{Z}_d$. This map is clearly surjective since for any class $c \mod d$ in $\mathbb{Z}_d$, we have $(1 \mod n) \otimes_{\mathbb{Z}} (c \mod m) \mapsto c \mod d$. Then we must show the map is injective. Note that any element of $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z}_m$ can be written in the form $1 \otimes b$, where we have started suppressing the mod notation. We have $1 \otimes b \mapsto 0 \in \mathbb{Z}_d$ if and only if $d \mid b$. Then it suffices to show that $1 \otimes d = 0$. By the Euclidean algorithm, there are integers a, b such that $d = an + bm$, so $1 \otimes d = 1 \otimes (an + bm) = 1 \otimes an + 1 \otimes bm = a(n \otimes 1) + b(1 \otimes m) = a(0 \otimes 1) + b(1 \otimes 0) = 0 + 0 = 0$. Thus if $d \mid b$, we have $1 \otimes b = (b/d)(1 \otimes d) = 0$, showing that the map defined above is injective, as desired. □

2 Problem 2

(a) Let $\{v_1, ..., v_n\}$ and $\{w_1, ..., w_m\}$ be bases for the vector spaces V and W over the field k respectively. Prove that $\{v_i \otimes w_j \mid i = 1, ..., n; j = 1, ..., m\}$ is a basis for $V \otimes_k W$.

Proof. As any element in $V \otimes_k W$ is a linear combination of pure tensors $v \otimes w$, we can show that $B = \{v_i \otimes w_j \mid i = 1, ..., n; j = 1, ..., m\}$ spans $V \otimes_k W$ by showing that B spans the set of pure tensors. For a pure tensor $v \otimes w$, we have $v = \sum_i a_i v_i$ and $w = \sum_j b_j w_j$, so that $v \otimes w = (\sum_i a_i v_i) \otimes (\sum_j b_j w_j) = \sum_{i,j} a_i b_j v_i \otimes w_j$, which is a linear combination of elements of B. Thus B spans $V \otimes_k W$. Note that this shows $\dim(V \otimes_k W) \leq \dim(V)\dim(W)$.

Let X be the free k-vector space spanned by the symbols $v_i w_j$, so that $\dim(X) = \dim(V)\dim(W)$. Then there is a bilinear map $V \times W \to X$ defined by sending $(v_i, w_j) \to v_i w_j$ and extending bilinearly. By the universal property of tensor product, there is a unique linear map $V \otimes_k W \to X$ which maps $v_i \otimes w_j \to v_i w_j$. Clearly this map is surjective, as it hits each basis element of X. This shows $\dim(V \otimes_k W) \geq \dim(V)\dim(W)$. Combined with the previously obtained inequality, we have $\dim(V \otimes_k W) = \dim(V)\dim(W)$. Since $V \otimes_k W$ is spanned by the set B of size $\dim(V)\dim(W)$, we have that B is a basis as desired. $\square$

(b) If $f : V \to V$ and $g : W \to W$ are k-linear transformations, prove that $\mathrm{Tr}(f \otimes g) = \mathrm{Tr}(f)\mathrm{Tr}(g)$.

Proof. With the result of part (a) in hand, we can write $f \otimes g$ as a matrix in terms of the basis $v_i \otimes w_j$ and compute the trace explicitly. Let $f(v_i) = \sum_j f_{ij} v_j$ and $g(w_i) = \sum_j g_{ij} w_j$ for $f_{ij}, g_{ij} \in k$. Then $(f \otimes g)(v_i \otimes w_j) = f(v_i) \otimes g(w_j) = (\sum_k f_{ik} v_k) \otimes (\sum_\ell g_{j\ell} w_\ell) = \sum_{k,\ell} f_{ik} g_{j\ell} v_k \otimes w_\ell$. In particular, the $v_i \otimes w_j$ component of $(f \otimes g)(v_i \otimes w_j)$ is $f_{ii} g_{jj}$. Thus $\mathrm{Tr}(f \otimes g) = \sum_{i,j} f_{ii} g_{jj} = \sum_i f_{ii} \sum_j g_{jj} = \sum_i f_{ii} \mathrm{Tr}(g) = \mathrm{Tr}(f)\mathrm{Tr}(g)$. $\square$

3 Problem 3

Let n be a positive integer and k be a field of characteristic p.

(a) If $p = 0$, show that $\mathbb{Z}_n \otimes_{\mathbb{Z}} k = 0$.

Proof. The multiplicative identity of $\mathbb{Z}_n \otimes_{\mathbb{Z}} k$ is $1 \otimes 1$. But $1 \otimes 1 = 1 \otimes (n \cdot \frac{1}{n}) = n \otimes \frac{1}{n} = 0 \otimes \frac{1}{n} = 0 \otimes 0$, which is the additive identity. Thus $\mathbb{Z}_n \otimes_{\mathbb{Z}} k = 0$ as desired. □

(b) If $p > 0$, determine $\mathbb{Z}_n \otimes_{\mathbb{Z}} k$.

Proof. If $p \nmid n$, then n is invertible in k, so the same proof as in part (a) shows that $\mathbb{Z}_n \otimes_{\mathbb{Z}} k = 0$.

If $p \mid n$, then we claim $\mathbb{Z}_n \otimes_{\mathbb{Z}} k \cong k$. We first give a simple proof if we accept the general fact that $(R/I) \otimes_R N \cong N/IN$ for I a two-sided ideal of a ring R and N a left R-module (which is proved in an example of Dummit and Foote, which I missed during several reads through the section). We then have $\mathbb{Z}_n \otimes_{\mathbb{Z}} k \cong k/(n\mathbb{Z})k = k/0 = k$, since $n = 0$ in k. For completeness, I give a more detailed proof below.

First consider a map $\mathbb{Z}_n \times k \to k$ which maps $(a \mod n, x)$ to ax. This is well-defined, since adding any multiple of n to a results in adding a multiple of p, which is 0 in k. It is also clearly $\mathbb{Z}$-bilinear. Thus we get a unique map $\mathbb{Z}_n \otimes_Z k \to k$ which maps $(a \mod n) \otimes x$ to ax. It is clearly surjective, since we can take $(1 \mod n) \otimes x$ as a preimage of any $x \in k$. To show this map is injective, notice first that any element in $\mathbb{Z}_n \otimes_{\mathbb{Z}} k$ can be written in the form $(1 \mod n) \otimes x$ for $x \in k$. This is a simple consequence of bilinearity (and written in more detail in my other solutions; I'm lazy). Then the kernel consists of those elements $(1 \mod n) \otimes x$ for which $1x = x = 0$; the kernel is 0 and the map is injective. Thus the map is an isomorphism and we are done. □

4 Problem 4

Let R be a ring and $f : N \to N'$ a left R-module map.

(a) If f is injective, prove that $R \otimes f : R \otimes_R N \to R \otimes_R N'$ is injective.

Proof. We first construct show that the map $R \otimes_R N \to N$ given by $r \otimes n \mapsto rn$ is an isomorphism. This map exists and is unique as defined, since $R \times N \to N, (r, n) \mapsto rn$ is bilinear. It is surjective because for any $n \in N$, we have $1 \otimes n \mapsto 1n = n$. Now note that any element in $R \otimes_R N$ can be written in the form $1 \otimes n$. By construction, any element is of the form $\sum_i r_i \otimes n_i$, and $\sum_i r_i \otimes n_i = \sum_i 1 \otimes r_i n_i = 1 \otimes (\sum_i r_i n_i)$. Then if an element is mapped to 0, it is of the form $1 \otimes n$ where $1n = n = 0$, so it is $1 \otimes 0 = 0$. Thus the map is an isomorphism. Since N is an arbitrary left R-module, we also have an isomorphism $R \otimes_R N' \cong N'$. The inverse map can be seen to be given by $n \mapsto 1 \otimes n$, since $rn \mapsto 1 \otimes rn = r \otimes n$.

Now, conjugating $R \otimes f$ by the isomorphisms gives a map $N \to N'$, which we prove is f. Indeed, we have $n \mapsto 1 \otimes n \mapsto 1 \otimes f(n) \mapsto 1f(n) = f(n)$. Thus we can write $R \otimes f$ as the composition of an isomorphism, followed by f, followed by another isomorphism. Since the composition of injective maps is injective, this proves $R \otimes f$ is injective. □

(b) Let M be a free right R-module of finite rank, say $M \cong R^n$. If f is injective, prove that $M \otimes f : M \otimes_R N \to M \otimes_R N'$ is injective.

Proof. Since $M \cong R^n = \bigoplus_i R$ and using Theorem 17 of Dummit and Foote Section 10.4, we have that $M \otimes f$ can be written as a composition of isomorphisms on the left and right of the map $\bigoplus_i (R \otimes_R N) \to \bigoplus_i (R \otimes_R N')$ which sends $(r_1 \otimes n_1, ...)$ to $(r_1 \otimes f(n_1), ...)$. In particular, this map is a direct sum of copies of the map $R \otimes f$, which is injective by part (a). A direct sum of injective maps is injective, since if $(a, b) \mapsto (g(a), h(b)) = (0, 0)$ where g, h are injective, then $a = 0, b = 0$. Thus $M \otimes f$ is equal to a composition of injective maps, so it is injective. □

5 Problem 5

Let R be a ring and S a subring of R. For any left S-module M and left R-module N, prove that $\mathrm{Hom}_R(R \otimes_S M, N) \cong \mathrm{Hom}_S(M, N)$ as abelian groups.

Proof. Given $f \in \mathrm{Hom}_S(M, N)$, define a map $\varphi_f : R \times M \to N$ which maps (r, m) to $rf(m)$. Then we have $\varphi_f(rs + r', m) = (rs + r')f(m) = rsf(m) + r'f(m) = rf(sm) + r'f(m) = \varphi_f(r.sm) + \varphi_f(r', m)$, so φ_f is S-balanced. Similarly $\varphi_f(sr, m) = srf(m) = s\varphi_f(m)$, so φ_f is S-bilinear. Thus there is a unique S-module morphism (which we lazily call) $\varphi_f : R \otimes_S M \to N$ which sends $r \otimes m$ to $rf(m)$. In fact, $\varphi_f(r'(r \otimes m)) = \varphi_f(r'r \otimes m) = r'rf(m) = r'\varphi_f(r \otimes m)$, so φ_f is an R-module morphism. Thus φ_- gives us a function from $\mathrm{Hom}_S(M, N)$ to $\mathrm{Hom}_R(R \otimes_S M, N)$, which we now argue is a group isomorphism.

Let $f, g \in \mathrm{Hom}_S(M, N)$. Then $\varphi_{f+g}(r \otimes m) = r(f + g)(m) = r(f(m) + g(m)) = rf(m) + rg(m) = \varphi_f(r \otimes m) + \varphi_f(r \otimes m)$. Since any element is a sum of pure tensors, this implies by linearity that $\varphi_{f+g} = \varphi_f + \varphi_g$. Thus φ_- is a group homomorphism. Now suppose $\varphi_f = 0$. In particular, $\varphi_f(1 \otimes m) = 1f(m) = f(m)$ is 0 for all m. Thus $f = 0$, so φ_- is injective. Finally, suppose $g \in \mathrm{Hom}_R(R \otimes_S M, N)$. Define function $f : M \to N$ by $f(m) = g(1 \otimes m)$. Then $g(r \otimes m) = g(r(1 \otimes m)) = rg(1 \otimes m) = rf(m)$, so if we can show that $f \in \mathrm{Hom}_S(M, N)$, we will be done. We have $f(sm + m') = g(1 \otimes (sm + m')) = g(1 \otimes sm + 1 \otimes m') = g(s \otimes m) + g(1 \otimes m') = sf(m) + f(m')$, so f is indeed an S-module morphism. Thus $g = \varphi_f$, so φ_- is surjective, implying φ_- is a group isomorphism. $\square$