MATH 7211 Homework 8

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1 Problem 10.1.18

Let $F = \mathbb{R}, V = \mathbb{R}^2$ and let T be the linear transformation from V to V which is rotation clockwise about the origin by $\pi/2$ radians. Show that V and 0 are the only $F[x]$-submodules for this T.

Proof. Of course, a module and 0 are always submodules of the same module. Dummit and Foote Section 10.1 gives a bijection between $F[x]$-submodules of V considered as an $F[x]$-module and T-stable linear subspaces of V. Now, 0 and V are the only 0 and 2-dimensional subspaces of V, and since the dimension of V is 2, there are no higher dimensional subspaces. Let W be any one-dimensional subspace of V, say spanned by $\begin{bmatrix} a \\ b \end{bmatrix}$. Then if $T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ -a \end{bmatrix}$ is in W, say $\begin{bmatrix} b \\ -a \end{bmatrix} = c \begin{bmatrix} a \\ b \end{bmatrix}$ for some $c \in F$, we find $a = -ac^2$. If $a \neq 0$, then $c^2 = -1$, which is impossible since $F = \mathbb{R}$. Thus $a = 0$. But then we have $\begin{bmatrix} b \\ 0 \end{bmatrix} = c \begin{bmatrix} 0 \\ b \end{bmatrix}$, which implies $b = 0$. But $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ does not span a one-dimensional space, which is a contradiction. Thus, none of the one-dimensional subspaces of V correspond to $F[x]$-submodules. Since we have exhausted all of the possible dimensions, we are done. $\square$

2 Problem 10.2.6

Prove that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$, where d is the greatest common divisor of m, n.

Proof. Let us first deal with the trivial cases of $n = 1$ and $m = 1$ (I don't know if these are supposed to be included, but might as well). For $n = 1$ we consider $\mathbb{Z}/n\mathbb{Z} = 0$ and $d = 1$. Thus we have to show $\text{Hom}(0, \mathbb{Z}/m\mathbb{Z}) = 0 = \text{Hom}(\mathbb{Z}/n\mathbb{Z}, 0)$. The first equality is true because the only morphism from 0 has to map to 0 in the codomain, and the second equality is true because the only map into 0 must be the one sending everything to 0.

Now assume that n, m are both greater than 1. A morphism from $\mathbb{Z}/n\mathbb{Z}$ is determined by where a generator, call it 1, is sent. However, not every element in $\mathbb{Z}/m\mathbb{Z}$ is a valid image of 1. In particular, we must have $nf(1) = f(n) = f(0) = 0$. Since we always have $mf(1) = 0$ for $f(1) \in \mathbb{Z}/m\mathbb{Z}$, the Euclidean algorithm implies $df(1) = 0$. Conversely, any element $x \in \mathbb{Z}/m\mathbb{Z}$ with $dx = 0$ satisfies $nx = (n/d)dx = (n/d)0 = 0$. Thus $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ is in bijection with the elements in $\mathbb{Z}/m\mathbb{Z}$ satisfying $dx = 0$. We have $x = m/d$ as one candidate for such elements. Conversely, if $dx = 0$, then m divides dx, so m/d divides x. Thus $0, m/d, ..., (d-1)m/d$ are the only elements satisfying $dx = 0$ in $\mathbb{Z}/m\mathbb{Z}$. In particular, the morphism $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ with $f(1) = m/d$ is a generator for all the other morphisms in $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$, since $kf(1) = km/d$. We see that f has order d, so $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$ as desired. □

3 Problem 10.3.9

Show that an R-module is irreducible iff $M \neq 0$ and M is cyclic module with any nonzero element as a generator. Determine all irreducible $\mathbb{Z}$-modules.

Proof. ($\rightarrow$) By definition, an irreducible module is non-zero. Let $a \in M$ be non-zero. Then Ra is a submodule of M. By irreducibility, it is either 0 or M. Since $a \neq 0$ and $1 \in R$, the submodule Ra cannot be 0. Thus $M = Ra$ for any non-zero $a \in M$ as desired.

($\leftarrow$) Let N be a non-zero submodule of M. Then N contains some non-zero element a. By assumption on M, we have $M = Ra$. But $Ra \subseteq N$ by definition of a module, so $N = M$, showing that M is irreducible.

Now we that we know irreducible modules are special non-zero cyclic modules, we can easily classify the irreducible $\mathbb{Z}$-modules. Of course, $\mathbb{Z}$-modules are just abelian groups under a different name. All cyclic groups are abelian, and the cyclic groups are $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ for $n > 1$. It is clear that $\mathbb{Z}$ is not irreducible, since the nonzero element 2 does not generate $\mathbb{Z}$; 1 is not a multiple of 2. For $n = p$ prime, we know that any non-zero element of $\mathbb{Z}/p\mathbb{Z}$ is a generator, since $|(\mathbb{Z}/p\mathbb{Z})^\times| = p - 1 = |\mathbb{Z}/p\mathbb{Z} - \{0\}|$. Thus all prime order cyclic groups are irreducible $\mathbb{Z}$-modules. If n is not prime, then choose a proper divisor $d > 1$ of n, so that d is of order n/d, i.e. it generates a subgroup of order $n/d < n$. Thus the irreducible $\mathbb{Z}$ modules are the prime order cyclic groups. $\square$

4 Problem 10.3.11

Show that if M_1 and M_2 are irreducible R-modules, then any nonzero R-module morphism from M_1 to M_2 is an isomorphism. Deduce that if M is irreducible then $\mathrm{End}_R(M)$ is a division ring.

Proof. Let $f : M_1 \to M_2$ be a non-zero R-module morphism. Then $\ker f$ is a submodule of M_1; by irreducibility, $\ker f$ is either 0 or M_1. If $\ker f = M_1$, then f is the zero map, which we assume it is not. Thus $\ker f = 0$, so f is injective. Similarly, $\mathrm{im} f$ is a submodule of M_2, so it is either 0 or M_2. If $\mathrm{im} f = 0$, that means f is the zero map, which we again assume it is not. Thus $\mathrm{im} f = M_2$, so f is surjective. Bijective module morphisms are isomorphisms, so f is an isomorphism.

Let M be irreducible. By the previous result, every non-zero element of $\mathrm{End}_R(M)$ is an isomorphism, i.e. there is an inverse element in $\mathrm{End}_R(M)$. Thus $\mathrm{End}_R(M)$ is a division ring. □

5 Problem 10.3.12

Let R be commutative and let A, B, and M be R-modules. Prove the following isomorphisms of R-modules:

1. $\mathrm{Hom}_R(A \times B, M) \cong \mathrm{Hom}_R(A, M) \times \mathrm{Hom}_R(B, M)$.

2. $\mathrm{Hom}_R(M, A \times B) \cong \mathrm{Hom}_R(M, A) \times \mathrm{Hom}_R(M, B)$.

Proof. 1. We define a function $\psi : \mathrm{Hom}_R(A \times B, M) \to \mathrm{Hom}_R(A, M) \times \mathrm{Hom}_R(B, M)$. We first need functions $i_1 : A \to A \times B$ and $i_2 : B \to A \times B$, which we can define via $a \mapsto (a, 0)$ and $b \mapsto (0, b)$. In fact, these are clearly module morphisms. Then we define $\psi(f) = (f \circ i_1, f \circ i_2)$. That ψ is a module morphism follows from composition being bilinear.

We can demonstrate an explicit inverse of ψ, namely $\psi^\wedge(g, h) : (a, b) \mapsto g(a) + h(b)$. Indeed, $(f \circ i_1)(a) + (f \circ i_2)(b) = f(a, 0) + f(0, b) = f(a, b)$, so $\psi^\wedge \circ \psi = \mathrm{id}$, and $(g(a) + h(0)) + (g(0) + h(b)) = g(a) + h(b)$, since $g(0) = h(0) = 0$, so $\psi \circ \psi^\wedge = \mathrm{id}$. Since ψ has an inverse function, it is a bijection; thus ψ is an isomorphism as desired.

2. The proof is similar; this time, we need morphisms $p_1 : A \times B \to A, p_2 : A \times B \to B$, which we take to be the projections onto each factor. Then we define $\varphi : \mathrm{Hom}_R(M, A \times B) \to \mathrm{Hom}_R(M, A) \times \mathrm{Hom}_R(M, B)$ by $\varphi(f) = (p_1 \circ f, p_2 \circ f)$. Once again, bilinearity of composition implies φ is a module morphism.

We prove φ is a bijection by giving an explicit inverse. Given $(g, h) \in \mathrm{Hom}_R(M, A) \times \mathrm{Hom}_R(M, B)$, define $(\varphi^\wedge(g, h))(m) = (g(m), h(m))$. Then $(\varphi \circ \varphi^\wedge)(g, h) = (p_1 \circ \varphi^\wedge(g, h), p_2 \circ \varphi^\wedge(g, h))$, and by definition of $\varphi^\wedge$, we have $p_1 \circ \varphi^\wedge(g, h) = g, p_2 \circ \varphi^\wedge(g, h) = h$, so $(\varphi \circ \varphi^\wedge)(g, h) = (g, h)$, i.e. $\varphi \circ \varphi^\wedge = \mathrm{id}$. We also see that $(\varphi^\wedge \circ \varphi)(f) = \varphi^\wedge(p_1 \circ f, p_2 \circ f)$. If $f(m) = (a, b)$, then $\varphi^\wedge(p_1 \circ f, p_2 \circ f)(m) = (p_1 \circ f(m), p_2 \circ f(m)) = (p_1(a, b), p_2(a, b)) = (a, b) = f(m)$. Thus $(\varphi^\wedge \circ \varphi)(f) = f$, so $\varphi^\wedge \circ \varphi = \mathrm{id}$. Since φ has an inverse function, it is a bijection, so it is an isomorphism, as desired. $\square$