MATH 7211 Homework 3

Andrea Bourque

May 9, 2023

1 Problem 13.3.1

Prove that it is impossible to construct the regular 9-gon.

Proof. If a regular 9-gon were constructible, then the internal angle $7(180)/9 = 140$ degrees would be constructible. Then its supplement, 40 degrees, would also be constructible. It is possible to construct angle bisectors, so then 20 degrees would be constructible. As is shown in Dummit and Foote Section 13.3 Theorem 24, and in class, $\alpha = 2\cos(20°)$ is a root of $f(x) = x^3 - 3x - 1$. This is an irreducible polynomial over $\mathbb{Q}$; since it is cubic, it suffices to show there are no rational roots. By the rational root theorem, the possible rational roots are ± 1, and $f(1) = -3, f(-1) = 1$, so $f(x)$ is irreducible as claimed. Thus $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$, which is not a power of 2, so α is not constructible. Thus the regular 9-gon is not constructible. $\square$

2 Problem 13.3.4

The construction of the regular 7-gon amounts to the constructibility of $\cos(2\pi/7)$. Use the fact that 2α is a root of $x^3 + x^2 - 2x - 1$ to prove that the regular 7-gon is not constructible.

Proof. We show $f(x) = x^3 + x^2 - 2x - 1$ is irreducible over the rationals. Since it is cubic, it suffices to show there are no rational roots. By the rational root theorem, the only possible rational roots are ± 1. We have $f(1) = -1$, $f(-1) = 1$, so $f(x)$ is irreducible. Thus $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$, which is not a power of 2, so α is not constructible. Thus the regular 7-gon is not constructible. $\square$

3 Problem 13.3.5

Use the fact that $\alpha = 2\cos(2\pi/5)$ satisfies the equation $x^2 + x - 1 = 0$ to conclude that the regular 5-gon is constructible.

Proof. Since α is the root of a quadratic polynomial over the rationals, we automatically know $[\mathbb{Q}(\alpha) : \mathbb{Q}] \leq 2$, which implies α is constructible. Then the angle $2\pi/5 = 72$ degrees is constructible. Since 72 degrees is the exterior angle of a regular 5-gon, a regular 5-gon is constructible. □

4 Problem 13.4.2

Determine the splitting field and its degree over $\mathbb{Q}$ for $x^4 + 2$.

Proof. The roots of $x^4 + 2$ in $\mathbb{C}$ are $\frac{1}{\sqrt[4]{2}}(\pm 1 \pm i)$, where the two $\pm$ signs are independent (i.e. all four combinations are allowed). Then the splitting field is contained in $\mathbb{Q}(\sqrt[4]{2}, i)$, since the roots can be written in terms of $\sqrt[4]{2}$ and i. To show that this field is actually the splitting field, we must do the reverse: Show that $\sqrt[4]{2}$ and i can be written in terms of the roots. We have

$$\frac{1}{\sqrt[4]{2}}(1+i) + \frac{1}{\sqrt[4]{2}}(1-i) = \frac{2}{\sqrt[4]{2}},$$

so

$$\sqrt[4]{2} = \frac{2}{\frac{1}{\sqrt[4]{2}}(1+i) + \frac{1}{\sqrt[4]{2}}(1-i)}.$$

We also have,

$$\frac{1}{\sqrt[4]{2}}(1+i) + \frac{1}{\sqrt[4]{2}}(-1+i) = \frac{2i}{\sqrt[4]{2}},$$

so

$$i = \frac{\frac{1}{\sqrt[4]{2}}(1+i) + \frac{1}{\sqrt[4]{2}}(-1+i)}{\frac{1}{\sqrt[4]{2}}(1+i) + \frac{1}{\sqrt[4]{2}}(1-i)}.$$

Thus the field obtained by adjoining the roots of x^4+2 to $\mathbb{Q}$ contains $\mathbb{Q}(\sqrt[4]{2}, i)$, while the field $\mathbb{Q}(\sqrt[4]{2}, i)$ contains all the roots of $x^4 + 2$. Thus $\mathbb{Q}(\sqrt[4]{2}, i)$ is the splitting field.

To find the degree, note that the subextension $\mathbb{Q}(\sqrt[4]{2})$ is contained in the reals, so $i \notin \mathbb{Q}(\sqrt[4]{2})$. Thus $[\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}(\sqrt{4}[2])] \neq 1$, so $[\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}(\sqrt{4}[2])] = 2$, since $x^2 + 1$ is irreducible over $\mathbb{Q}(\sqrt{4}[2])$. Next, $x^4 - 2$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion applied with the prime 2. Thus $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 4$. Therefore, $[\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}] = 8$. $\square$

5 Problem 13.4.4

Determine the splitting field and its degree over $\mathbb{Q}$ for $x^6 - 4$.

Proof. Let $\omega = \exp(2\pi i/6)$. Clearly $\omega \notin \mathbb{Q}$, since ω has a nonzero imaginary part $\sin(2\pi/6) = \sqrt{3}/2$. Note that $\omega^3 = \exp(\pi i) = -1$, so ω is a root of $x^3 + 1 = (x+1)(x^2 - x + 1)$. Since $\omega \neq -1$, we have ω is a root of $x^2 - x + 1$. Since $\omega \notin \mathbb{Q}$, this quadratic must be the minimal polynomial of ω over $\mathbb{Q}$. Thus, $[\mathbb{Q}(\omega) : \mathbb{Q}] = 2$.

Now, the roots of $x^6 - 4$ are $\sqrt[3]{2}\omega^k$ for $k \in \{0, 1, 2, 3, 4, 5\}$. Thus, $\mathbb{Q}(\sqrt[3]{2}, \omega)$ contains all the roots. Furthermore, since $\sqrt[3]{2}$ is a root, and $\omega = \sqrt[3]{2}\omega/\sqrt[3]{2}$ can be written in terms of the roots, $\mathbb{Q}(\sqrt[3]{2}, \omega)$ must be the splitting field.

As we have already noted, $[\mathbb{Q}(\omega) : \mathbb{Q}] = 2$. Since $f(x) = x^3 - 2$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion with the prime 2, $[\mathbb{Q}(\sqrt[3]{2} : \mathbb{Q}] = 3$. Since $[\mathbb{Q}(\omega) : \mathbb{Q}]$ and $[\mathbb{Q}(\sqrt[3]{2} : \mathbb{Q}]$ are coprime, the degree $[\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}]$ is the product of the degrees, i.e. 6. □