MATH 7211 Homework 13

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1 Setup for Problems 18.3.16 and 18.3.17

Let F be the field of algebraic numbers. Let G be a finite group. Let $\varphi : G \to GL_m(F)$ be any representation with character ψ. Let $\sigma \in \mathrm{Aut}(F/\mathbb{Q})$. The map $\varphi^\sigma : G \to GL_m(F)$ is defined by applying σ to the entries of matrices in the image.

2 Problem 18.3.16

Prove that φ^σ is a representation. Prove also that the character of φ^σ is $\sigma \circ \psi$.

Proof. Let $g, h \in G$. Then the i, j entry of $\varphi^\sigma(gh)$, denoted $\varphi^\sigma(gh)_{ij}$, is $\sigma(\varphi(gh)_{ij})$, by definition. We know that $\varphi(gh) = \varphi(g)\varphi(h)$, so $\varphi(gh)_{ij} = \sum_k (\varphi(g)_{ik}\varphi(h)_{kj})$. Then

$$\begin{aligned}\varphi^\sigma(gh)_{ij} = \sigma(\sum_k (\varphi(g)_{ik}\varphi(h)_{kj})) &= \sum_k (\sigma(\varphi(g)_{ik})\sigma(\varphi(h)_{kj})) \\ &= \sum_k (\varphi^\sigma(g)_{ik}\varphi^\sigma(h)_{kj}).\end{aligned}$$

Thus $\varphi^\sigma(gh) = \varphi^\sigma(g)\varphi^\sigma(h)$, as desired. For the character, we see that the trace of $\varphi^\sigma(g)$ is

$$\sum_i \varphi^\sigma(g)_{ii} = \sum_i \sigma(\varphi(g)_{ii}) = \sigma(\sum_i \varphi(g)_{ii}) = \sigma(\psi(g)),$$

so that the character of φ^σ is indeed $\sigma \circ \psi$. □

3 Problem 18.3.17

Prove that φ is irreducible if and only if φ^σ is irreducible.

Proof. Recall that a representation is irreducible iff its character has norm 1. If G has r conjugacy classes of size $d_1, ..., d_r$ with representatives $g_1, ..., g_r$, then we can write the norm squared of a character ψ as $\frac{1}{|G|}\sum_{i=1}^r(d_i\psi(g_i)\psi(g_i^{-1}))$. As we saw from the previous problem, the character of φ^σ is $\sigma \circ \psi$. Thus the norm squared is

$$\frac{1}{|G|}\sum_{i=1}^{r}(d_i\sigma(\psi(g_i))\sigma(\psi(g_i^{-1}))) = \sigma\left(\frac{1}{|G|}\sum_{i=1}^{r}(d_i\psi(g_i)\psi(g_i^{-1}))\right),$$

using the fact that σ is a field automorphism which fixes $\mathbb{Q}$. Furthermore, since σ fixes $\mathbb{Q}$, we see that the norm squared of $\sigma \circ \psi$ is 1 iff the norm squared of ψ is 1, which is equivalent to the statement of the problem. □

4 Problem 19.1.6

Calculate the character table of A_5.

Proof. We begin by stating (without proof) that the conjugacy classes of A_5 are given by representatives $1, (123), (12)(34), (12345), (21345)$, with corresponding sizes $1, 20, 15, 12, 12$. We then have five irreducible characters, starting with the trivial character χ_1.

It is actually possible to determine the degrees of the other characters, as follows. We know that the degrees are divisors of $|A_5| = 60$ and that the sum of their squares is $|A_5| = 60$. One of the degrees is 1, so we have four divisors of 60, say $2 \leq a \leq b \leq c \leq d$, which satisfy $a^2 + b^2 + c^2 + d^2 = 59$. Clearly $d < \sqrt{59}$, so the biggest possible value is $d = 6$. We then get $a^2 + b^2 + c^2 = 23$. The biggest possible value of c is 4, which leads to $a^2 + b^2 = 7$. This clearly has no solutions. Moving down to $c = 3$ gives $a^2 + b^2 = 14$, which has no solutions. Moving to $c \leq 2$ gives $a^2 + b^2 \geq 19$, but $a^2 + b^2 \leq 2c^2 \leq 8$. Thus $d = 6$ gives no solutions.

The next biggest possibility is $d = 5$. This will end up giving a solution, so let us first rule out $d \leq 4$. For $d = 4$ we have $a^2 + b^2 + c^2 = 43$. The biggest choice of c is 4, giving $a^2 + b^2 = 27$, which quickly can be seen to have no solution. For $c \leq 3$, we have $a^2 + b^2 \geq 34$, but $a^2 + b^2 \leq 2c^2 \leq 18$. Thus $d = 4$ does not work. For $d \leq 3$, we have $a^2 + b^2 + c^2 \geq 50$, but $a^2 + b^2 + c^2 \leq 3d^2 \leq 27$. Thus there are no solutions for $d \leq 4$.

Let us return to $d = 5$. We have $a^2 + b^2 + c^2 = 34$. For $c \leq 3$, we have $a^2 + b^2 \geq 25$, but $a^2 + b^2 \leq 2c^2 \leq 18$. Thus the only option is $c = 4$, for which $a^2 + b^2 = 18$. Then the only option is $a = b = 3$. It is very important to note that we have gotten the *only* possible integer solution to four divisors of 60 whose sum of squares gives 59. These must be the degrees of our irreducible representations. We therefore have the information in Table 1.

Perhaps this was overkill, but now we now what to look for. Consider the character table of S_5 in Dummit and Foote Section 19.1. The character χ_3 from that table (which we will call χ_3' to avoid confusion) restricts to a character χ reading $4, 1, 0, -1, -1$. Recall that a character is irreducible iff its norm (squared) is 1. Using this, we can check that this χ is irreducible, and so is our χ_4. Indeed,

$$(\chi, \chi) = \frac{1}{60}(1 \cdot 4^2 + 20 \cdot 1^2 + 15 \cdot 0^2 + 12 \cdot (-1)^2 + 12 \cdot (-1)^2) = 1.$$

Thus we have the information in Table 2.

The exact same argument proceeds to show that the character χ_5' of S_5 restricts to an irreducible degree 5 character of A_5. It reads $5, -1, 1, 0, 0$, and we can use inner product computation to show that it is irreducible. This gives us our character χ_5, so we have the information in Table 3.

Now, let us restrict the remaining character χ_7' of S_5. We get a character χ of A_5 reading $6, 0, -2, 1, 1$. This is not irreducible; its norm squared is 2. However, this implies it is a sum of two distinct irreducibles. In particular, by looking at its degree, we see that it is either $\chi_1+\chi_5$ or $\chi_2+\chi_3$. We can rule out the first case by showing that $(\chi, \chi_1) = 0$, which is a direct computation. Thus $\chi_2 + \chi_3$ reads $6, 0, -2, 1, 1$. Let χ_2 read $3, w, x, y, z$. Then χ_3 reads $3, -w, -2-x, 1-y, 1-z$. We have the information in Table 4.

We know $(\chi_2, \chi_1) = (\chi_2, \chi_4) = (\chi_2, \chi_5) = 0$. This gives the equations

$$\begin{aligned} 3 + 20w + 15x + 12y + 12z &= 0, \\ 12 + 20w - 12y - 12z &= 0, \\ 15 - 20w + 15x &= 0. \end{aligned}$$

Adding the first two equations gives $15 + 40w + 15x = 0$, so comparing to the third equation gives $w = 0$. Plugging this back in gives $x = -1$ and $y + z = 1$. Thus we have the information in Table 5.

All that remains is to compute y. Orthonormality $(\chi_2, \chi_2) = 1$ tells us that $|y|^2 + |1-y|^2 = 3$, and if we knew that $y \in \mathbb{R}$, we could conclude. It is possible to deduce $y \in \mathbb{R}$. Recall $\overline{\chi(g)} = \chi(g^{-1})$ for arbitrary character χ of any group. If we can show that $(12345)^{-1}$ is conjugate to (12345) in A_5, then we can conclude that $\overline{y} = y$, so that y is real. Indeed, we have

$$(12345)^{-1} = (15432) = (25)(34)(12345)(25)(34),$$

so y is real. Thus $y^2 + (1-y)^2 = 3$ is a quadratic equation that gives two solutions $y = \dfrac{1 \pm \sqrt{5}}{2}$. Since the two solutions add to 1, it doesn't matter which we take as y, due to the symmetry in our table. Thus, taking y to be the positive solution, we have the completed character table of A_5 in Table 6. $\square$

Table 1:

class	1	(123)	(12)(34)	(12345)	(21345)
size	1	20	15	12	12
χ_1	1	1	1	1	1
χ_2	3	?	?	?	?
χ_3	3	?	?	?	?
χ_4	4	?	?	?	?
χ_5	5	?	?	?	?

Table 2:

class	1	(123)	(12)(34)	(12345)	(21345)
size	1	20	15	12	12
χ_1	1	1	1	1	1
χ_2	3	?	?	?	?
χ_3	3	?	?	?	?
χ_4	4	1	0	-1	-1
χ_5	5	?	?	?	?

Table 3:

class	1	(123)	(12)(34)	(12345)	(21345)
size	1	20	15	12	12
χ_1	1	1	1	1	1
χ_2	3	?	?	?	?
χ_3	3	?	?	?	?
χ_4	4	1	0	-1	-1
χ_5	5	-1	1	0	0

Table 4:

class	1	(123)	(12)(34)	(12345)	(21345)
size	1	20	15	12	12
χ_1	1	1	1	1	1
χ_2	3	w	x	y	z
χ_3	3	$-w$	$-2-x$	$1-y$	$1-z$
χ_4	4	1	0	-1	-1
χ_5	5	-1	1	0	0

Table 5:

class	1	(123)	(12)(34)	(12345)	(21345)
size	1	20	15	12	12
χ_1	1	1	1	1	1
χ_2	3	0	-1	y	$1-y$
χ_3	3	0	-1	$1-y$	y
χ_4	4	1	0	-1	-1
χ_5	5	-1	1	0	0

Table 6:

class	1	(123)	(12)(34)	(12345)	(21345)
size	1	20	15	12	12
χ_1	1	1	1	1	1
χ_2	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_3	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_4	4	1	0	-1	-1
χ_5	5	-1	1	0	0

5 Problem 19.1.7

Show that S_6 has an irreducible character of degree 5.

Proof. As usual, we have a character of S_6 given by the action of S_6 on a 6 element set. Also, as usual, it is the sum of the trivial character and an irreducible character, as we will show below. The conjugacy classes of S_6 are given by the cycle types, which correspond to the partitions of 6. There are 11 of these, with representatives

$$1,(12),(123),(1234),(12345),(123456),(12)(34),$$
$$(12)(34)(56),(123)(45),(123)(456),(1234)(56).$$

The corresponding sizes of the conjugacy classes can be computed via the following combinatorial formula (from planetmath.org/conjugacyclassesinthesymmetricgroupsn): consider a cycle type and let $m_1,...,m_r$ be the distinct lengths of cycles appearing in it. Let k_i be the number of cycles of length m_i in the given cycle type. Then the size of the conjugacy class corresponding to the cycle type is $\frac{6!}{\prod_i (k_i! m_i^{k_i})}$. Using this formula, we compute the sizes of conjugacy classes in S_6 (in order of the cycle types I wrote above):

$$1,15,40,90,144,120,45,$$
$$15,120,40,90.$$

We must also compute the character on each conjugacy class. This is given by the number of fixed points of each representative (i.e. the number of length 1 cycles in the cycle type), and given as follows:

$$6,4,3,2,1,0,2,$$
$$0,1,0,0.$$

Now, we subtract the trivial character, giving a character with values

$$5,3,2,1,0,-1,1,$$
$$-1,0,-1,-1.$$

We may compute the norm squared of this character via the formula $\frac{1}{720}\sum_{i=1}^{11} d_i|\chi(g_i)|^2$, where d_i is the size of the ith conjugacy class, and g_i is a representative for the ith conjugacy class. Doing so gives 1. A character is irreducible iff its norm (squared) is 1, so we have that this character is irreducible. Furthermore, the degree of a character can be seen as its output on the trivial conjugacy class, which in our case is 5. Thus, we have demonstrated an irreducible character of S_6 with degree 5. □

6 Problem 19.1.8

Calculate the character table of D_{10}.

Proof. Recall $D_{10} = \langle r, s \mid r^5 = s^2 = srsr = e \rangle$. Also recall that the elements can be written uniquely in the form $s^i r^j$ for $i \in \{0, 1\}$ and $j \in \{0, 1, 2, 3, 4\}$. The conjugacy classes are $\{e\}$, $\{r, r^4\}$, $\{r^2, r^3\}$, and $\{s, sr, sr^2, sr^3, sr^4\}$. Thus, there will be four irreducible characters. We obviously have the trivial character to start. Recall that the sum of the squares of the degrees of the irreducible representations equals the order of the group, and that each degree is a divisor of the order of the group. Then we have three divisors of 10, say $a \leq b \leq c$, satisfying $a^2 + b^2 + c^2 = 9$. It is clear that the only possibility is $a = 1, b = 2, c = 2$, since $c \geq 3$ forces a and b to be non-positive. Thus we have the information in Table 6.

Let us determine what the second degree 1 character, χ_2, is. The element r is represented by a complex number satisfying $x^5 = 1$, and s is represented by a complex number satisfying $y^2 = 1$. Furthermore, the group relation $srsr = e$ implies that $yxyx = 1$, so $x^2 = 1$. If $x^2 = x^5 = 1$, then $x = 1$. Thus r must act trivially. This determines $\chi_2(r), \chi_2(r^2)$. So far, χ_2 looks exactly like χ_1. Therefore, the only way to make a different representation is to require that s does not act trivially. The only other option is $s = -1$, so $\chi_2(s) = -1$. Thus we have the information in Table 6.

To get information on χ_3, we apply the orthogonality relations $(\chi_3, \chi_1) = 0 = (\chi_3, \chi_2)$ to get $2 + 2\chi_3(r) + 2\chi_3(r^2) \pm 5\chi_3(s) = 0$. Thus we see $\chi_3(s) = 0$ and $\chi_3(r) + \chi_3(r^2) = -1$. Note also that the exact same equations hold for χ_4. Let $\chi_3(r) = X$ and $\chi_3(r) = Y$. Using column orthogonality for columns e and r, we have $1 + 1 + 2X + 2Y = 0$, so $Y = -1 - X$. Thus, we have the information in Table 6.

It remains to determine X. Orthonormality $(\chi_3, \chi_3) = 1$ tells us that $|X|^2 + |-1 - X|^2 = 3$. We can deduce that X is real. Indeed, $\overline{X} = \overline{\chi_3(r)} = \chi_3(r^{-1}) = \chi_3(r^4) = \chi_3(r) = X$, since $\{r, r^4\}$ is a conjugacy class in D_{10}. Thus $X^2 + (-1 - X)^2 = 3$, which gives two solutions $X = \dfrac{-1 \pm \sqrt{5}}{2}$. Since the solutions add to -1, it doesn't matter which one we take as X, thanks to the inherent symmetry between χ_3 and χ_4. Thus we take the positive solution. The completed character table is in Table 6. □

Table 7:

class	e	r	r^2	s
size	1	2	2	5
χ_1	1	1	1	1
χ_2	1	?	?	?
χ_3	2	?	?	?
χ_4	2	?	?	?

Table 8:

class	e	r	r^2	s
size	1	2	2	5
χ_1	1	1	1	1
χ_2	1	1	1	-1
χ_3	2	?	?	?
χ_4	2	?	?	?

Table 9:

class	e	r	r^2	s
size	1	2	2	5
χ_1	1	1	1	1
χ_2	1	1	1	-1
χ_3	2	X	$-1-X$	0
χ_4	2	$-1-X$	X	0

Table 10:

class	e	r	r^2	s
size	1	2	2	5
χ_1	1	1	1	1
χ_2	1	1	1	-1
χ_3	2	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	0
χ_4	2	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	0