MATH 7211 Homework 11

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1 Problem 18.1.2

Let $\varphi : G \to GL_n(F)$ be a matrix representation. Prove that the map $g \mapsto \det(\varphi(g))$ is a degree 1 representation.

Proof. From linear algebra, we know that $\det(AB) = \det(A)\det(B)$ for any two n by n matrices A, B. Furthermore, $A \in GL_n(F)$ means $\det(A) \in F$ is invertible, so that $\det : GL_n(F) \to F^\times$ is a homomorphism. Since compositions of homomorphisms is a homomorphism, and $F^\times$ is the same thing as $GL_1(F)$, the map $\det \circ \varphi : G \to GL_1(F)$ is a homomorphism, i.e. it is a degree 1 representation of G. □

2 Problem 18.1.3

Prove that the degree 1 representations of G are in bijection with the degree 1 representations of $G/[G,G]$.

Proof. Let $\varphi : G \to GL(V)$ be a degree 1 representation of G, so V is a 1-dimensional F-vector space. Let $a, b \in G$. Then $\varphi([a,b]) = \varphi(a)\varphi(b)\varphi(a)^{-1}\varphi(b)^{-1} = 1$, since anything in $\text{im}\varphi \subset GL(V) = F^\times$ commutes. Since any commutator is in the kernel of φ and commutators generate $[G,G]$, the whole of $[G,G]$ must be in $\ker\varphi$. Then we have a well-defined quotient homomorphism $\bar{\varphi} : G/[G,G] \to GL(V)$ defined by $\bar{\varphi}(g[G,G]) = \varphi(g)$, i.e. a degree 1 representation of $G/[G,G]$. Conversely, given a degree 1 representation ψ of $G/[G,G]$, let $\tilde{\psi} = \psi \circ \pi$, where $\pi : G \to G/[G,G]$ is the quotient homomorphism. Then $\tilde{\psi}$ is a homomorphism $G \to GL(V)$, so it is a degree 1 representation. To show that these processes are inverses, we must show that $\varphi = \overline{\varphi} \circ \pi$ and $\psi = \overline{\psi \circ \pi}$. This is clear from the definitions:

$$(\overline{\varphi} \circ \pi)(g) = \overline{\varphi}(g[G,G]) := \varphi(g),$$
$$(\overline{\psi \circ \pi})(g[G,G]) := (\psi \circ \pi)(g) = \psi(g[G,G]).$$

Thus we have a bijection as desired. □

3 Problem 18.1.7

Let V be the 4-dimensional permutation module for S_4. Let $\pi : D_8 \to S_4$ be the permutation representation of D_8 obtained from left multiplication on left cosets of $\langle s \rangle$. Make V into an FD_8-module via π and write out the 4×4 matrices for r and s given by this representation with respect to $e_1, ..., e_4$.

Proof. The left cosets may be written $S_1 = \{e, s\}, S_2 = \{r, rs\}, S_3 = \{r^2, r^2 s\}, S_4 = \{r^3, r^3 s\}$. As in the text of Dummit and Foote, we can make V an FD_8-module by $g \cdot e_i = e_{\pi(g)(i)}$. To write the matrices for r and s, it suffices to compute $\pi(r)$ and $\pi(s)$. To compute these permutations, we must compute rS_i and sS_i for each i. The trivial case is $rS_i = S_{i+1}$ for $i = 1, 2, 3$, and $rS_4 = S_1$, where at the end we use $r^4 = e$. The computation that requires more work is for sS_i. Clearly $sS_1 = S_1$. Then, $sS_2 = \{sr, srs\}$. Using the relation $rsr = s$, we have that $sr = r^3 s$, so $sS_2 = S_4$. Next, $sS_3 = \{sr^2, sr^2 s\}$. Using $rsr = s$ again, we see that $r^2 sr^2 = r(rsr)r = rsr = s$, so $r^2 s = sr^2$. Thus $sS_3 = S_3$. Finally, $sS_4 = \{sr^3, sr^3 s\} = S_2$, since as we saw before, $r^3 s = sr$, so $sr^3 s = r$.

In summary, $\pi(r)$ is the cycle (1234) and $\pi(s)$ is the transposition (24). As matrices, we have

$$r = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

□

4 Problem 18.1.8

Let V be the FS_n-module described in Examples 3 and 10.

(a) Prove that if $v \in V$ is fixed by S_n, then v is an F-multiple of $e_1 + e_2 + ... + e_n$.

Proof. Write $v = a_1e_1 + ... + a_ne_n$. Since v is fixed by the transposition $(1k)$ for each $k = 2, ..., n$, we have that $(a_1, ..., a_k, ..., a_n) = (a_k, ..., a_1, ..., a_n)$, so $a_1 = a_k$ for $k = 2, ..., n$. Thus, $v = a_1(e_1 + ... + e_n)$ as desired. □

(b) Prove that if $n \geq 3$, then V has a unique 1-dimensional submodule, namely the $N = \text{span}\ (e_1 + e_2 + ... + e_n)$.

Proof. Suppose that the span of $v \in V$ is a 1-dimensional submodule of V. Write $v = a_1e_1 + ... + a_ne_n$. For the span of v to be 1-dimensional, v must be non-zero, so there is some a_k which is non-zero. If some $a_i = 0$, then applying the transposition (ik) gives a multiple of v with zero e_k component. This is only possible if $(ik) \cdot v = 0$, but we know that $(ik) \cdot v$ is non-zero, since it has a non-zero e_i component. Therefore, all the coefficients of v are non-zero. Since $n \geq 3$, we can look at the transpositions $(2k)$ for $k = 3, ..., n$. We have $(a_1, a_k, ..., a_2, ..., a_n)$ is a multiple of $(a_1, a_2, ..., a_k, ..., a_n)$. Since $a_1 \neq 0$, we then have $(a_1, a_2, ..., a_k, ..., a_n) = (a_1, a_k, ..., a_2, ..., a_n)$, or $a_2 = a_k$. Thus $a_2 = a_3 = ... = a_n$. In a similar manner, using the fact that $a_n \neq 0$, we can apply the transposition (12) to obtain $a_1 = a_2$. Thus $v = a_1(e_1 + ... + e_n)$. Since we started with an arbitrary 1-dimensional submodule of V and showed that it is spanned by (a multiple of) $e_1 + ... + e_n$, it follows that the span of $e_1 + ... + e_n$ is the unique 1-dimensional submodule. □

5 Problem 18.1.11

Let $\varphi : S_n \to GL_n(F)$ be the matrix representation given by the permutation matrices. Prove that $\det(\varphi(\sigma)) = \operatorname{sgn}(\sigma)$ for all $\sigma \in S_n$. (check on transpositions).

Proof. Since S_n is generated by transpositions, it suffices to show this for the case where σ is a transposition, since $\det \circ \varphi$ and sgn are homomorphisms, and homomorphisms are determined by their images on generators. We know that the sign of a transposition is, by definition, -1. So, we must show that $\det(\varphi(\sigma)) = -1$ if σ is a transposition. This is obvious if we accept the definition of det in terms of alternating bilinear map on columns; then $\det(\varphi(\sigma)) = -\det(I_n) = -1$. For completeness, I will give a proof which uses induction and computing the determinant via minors.

The only transposition matrix in $n = 2$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which clearly has determinant -1. Now let $n > 2$. Suppose σ swaps i and j. For $k \neq i, j$, we must have $(\varphi(\sigma))_{kk} = 1$, and, futhermore, this is the only non-zero entry in row k. Then we can expand the determinant along row k, giving that $\det \varphi(\sigma) = \det B$, where B is the corresponding minor of $\varphi(\sigma)$. Since both row and column k only have non-zero entry at (k, k), the minor B keeps all of the other non-zero entries of $\varphi(\sigma)$. In particular, B is a permutation matrix for S_{n-1}, so by induction, $\det B = -1$, and so, $\det \varphi(\sigma) = -1$ as desired. □