MATH 7210 Homework 6

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1 Problem 1

Let $p > 3$ be a prime congruent to 3 mod 4. Show that there are exactly four groups of order $4p$ up to isomorphism.

Proof. By the Sylow theorems, we have $n_2 = 1$ or p, and $n_p = 1$, since $2, 4 \not\equiv 1 \mod p$. Thus a group of order $4p$ has a normal subgroup of order p; this is $\mathbb{Z}/p\mathbb{Z}$ up to isomorphism.

If $n_2 = 1$, then the group is a direct product of $\mathbb{Z}/p\mathbb{Z}$ with a group of order 4. There are exactly two groups of order 4 up to isomorphism, namely $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. So in the case $n_2 = 1$, we have the groups $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

If $n_2 = p$, we must consider semidirect products. Let $H = \mathbb{Z}/p\mathbb{Z}$. Then $\mathrm{Aut}(H) \cong \mathbb{Z}/(p-1)\mathbb{Z}$. Since $p \equiv 3 \mod 4$, there are no automorphisms of order 4. Furthermore, a non trivial image of the groups $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ must be an order 2 subgroup in $\mathrm{Aut}(H)$ by Lagrange's theorem. Again, since $p \equiv 3 \mod 4$, this means that the image will be a 2-Sylow subgroup, which are all conjugate. By Problem 5.5.6, this means that regardless of the choice of $\varphi : K \to \mathrm{Aut}(H)$, where K is either $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the semidirect products will be isomorphic. Thus there are two more groups of order $4p$ up to isomorphism, $\mathbb{Z}/p\mathbb{Z} \rtimes_\varphi (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ and $\mathbb{Z}/p\mathbb{Z} \rtimes_\varphi \mathbb{Z}/4\mathbb{Z}$, where φ is a nontrivial homomorphism in each case. $\square$

2 Problem 2

Let p be a prime congruent to 1 mod 4. Show that there are exactly five groups of order $4p$ up to isomorphism.

Proof. The analysis in the beginning is the same as in Problem 1: when $n_2 = 1$, we have the groups $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

When $n_2 = p$, we again consider semidirect products. This time, with $H = \mathbb{Z}/p\mathbb{Z}$, $\mathrm{Aut}(H) \cong \mathbb{Z}/(p-1)\mathbb{Z}$ has an order of $p-1$ which is divisible by 4. Let $\mathrm{Aut}(H) = \langle a \rangle$, where $a \in (\mathbb{Z}/p\mathbb{Z})^*$. Then $1 \equiv a^{p-1} \equiv (a^{(p-1)/4})^4 \bmod p$. Since a is a generator of order exactly $p-1$, none of $a^{(p-1)/4}, a^{2(p-1)/4}, a^{3(p-1)/4}$ can be identity. Thus $a^{(p-1)/4}$ is an element of order 4 in $\mathrm{Aut}(H)$. Therefore, we can form a homormorphism $\mathbb{Z}/4\mathbb{Z} \to \mathrm{Aut}(H)$ which in which a generator of $\mathbb{Z}/4\mathbb{Z}$ is sent to an order 4 automorphism, which was not possible in the $p \equiv 3$ mod 4 case. This gives one extra group for a total of five. $\square$

3 Problem 5.2.7

Let p be prime and let $A = \langle x_1 \rangle \times ... \times \langle x_n \rangle$ be an abelian p-group, where $|x_i| = p^{\alpha_i} > 1$ for all i. Let $\varphi : A \to A$ be $\varphi(x) = x^p$.

a) Prove that φ is a homomorphism.

Proof. Since A is abelian, $(xy)^n = x^n y^n$ for all $x, y \in A$, $n \in \mathbb{N}$. Thus $\varphi(xy) = (xy)^p = x^p y^p = \varphi(x)\varphi(y)$. □

b) Decribe the image and kernel of φ in terms of the given generators.

Proof. The kernel of φ consists of tuples where each entry is an order p element in the corresponding cyclic group. The order p elements in $\langle x_i \rangle$ are generated by $x_i^{p^{\alpha_i - 1}}$, so $\ker \varphi = \langle x_1^{p^{\alpha_1 - 1}} \rangle \times ... \times \langle x_n^{p^{\alpha_n - 1}} \rangle$.

Since each generator x_i gets mapped to x_i^p, it follows that $\mathrm{im}\varphi = \langle x_1^p \rangle \times ... \times \langle x_n^p \rangle$ □

c) Prove that $\ker \varphi$ and $A/\mathrm{im}\varphi$ both have rank n, and furthermore are isomorphic to the elementary abelian group of order p^n.

Proof. From $\ker \varphi = \langle x_1^{p^{\alpha_1 - 1}} \rangle \times ... \times \langle x_n^{p^{\alpha_n - 1}} \rangle$, we clearly see that $\ker \varphi$ has rank n, and since there are n generators of order p, consisting of $(..., x_i^{p^{\alpha_i - 1}}, ...)$ with identity elements everywhere except one spot, this is isomorphic to E_{p^n}.

From $\mathrm{im}\varphi = \langle x_1^p \rangle \times ... \times \langle x_n^p \rangle$, we see that in $A/\mathrm{im}\varphi$, each cyclic group is reduced to order p, since the p powers vanish in the quotient. Explicitly, $A/\mathrm{im}\varphi$ is generated by $(..., x_i, ...)\mathrm{im}\varphi$, where there are identity elements everywhere except one spot, and since $(..., x_i, ...)^p = (..., x_i^p, ...) \in \mathrm{im}\varphi$, the generators have order p. Thus $A/\mathrm{im}\varphi \cong E_{p^n}$. □

4 Problem 5.5.6

Let K be cyclic, H a group, and φ_1, φ_2 are homomorphisms $K \to \mathrm{Aut}(H)$ such that $\varphi_1(K)$ and $\varphi_2(K)$ are conjugate subgroups of $\mathrm{Aut}(H)$. If K is infinite, assume φ_1, φ_2 are injective. Prove by constructing an explicit isomorphism that $H \rtimes_{\varphi_1} K \cong H \rtimes_{\varphi_2} K$.

Proof. Suppose $\sigma\varphi_1(K)\sigma^{-1} = \varphi_2(K)$, for some $\sigma \in \mathrm{Aut}(H)$. Since K is cyclic, for some $a \in \mathbb{Z}$, we have $\sigma\varphi_1(k)\sigma^{-1} = \varphi_2(k)^a$ for all $k \in K$. Let $\psi : H \rtimes_{\varphi_1} K \to H \rtimes_{\varphi_2} K$ be $\psi((h,k)) = (\sigma(h), k^a)$. We have

$$\begin{aligned}
&\psi((h_1,k_1)(h_2,k_2)) = \psi(h_1\varphi_1(k_1)(h_2), k_1k_2) \\
&= (\sigma(h_1)\sigma\varphi_1(k_1)(h_2), (k_1k_2)^a) = (\sigma(h_1)\varphi_2(k_1)^a\sigma(h_2), k_1^a k_2^a) \\
&= (\sigma(h_1)\varphi_2(k_1^a)\sigma(h_2), k_1^a k_2^a) = (\sigma(h_1), k_1^a)(\sigma(h_2), k_2^a) \\
&= \psi(h_1,k_1)\psi(h_2,k_2).
\end{aligned}$$

Thus ψ is a homomorphism.

First suppose K is finite. In order for $a \in \mathbb{Z}$ to preserve the image, we must have a coprime to $|K|$, so that $x \mapsto x^a$ is an automorphism of K. Then a has an inverse b mod $|K|$. Let $\xi((h,k)) = (\sigma^{-1}(h), k^b)$. Since $ab \equiv 1 \bmod |K|$, $k^{ab} = k$ for all k. Then it is clear that $\xi \circ \psi = \psi \circ \xi = \mathrm{id}$, since $(k^a)^b = (k^b)^a = k$.

Now suppose K is infinite. Then the only a which could preserve the image is $a = \pm 1$, so that $a^2 = 1$. It follows that $\xi((h,k)) = (\sigma^{-1}(h), k^a)$ is an inverse for ψ, since $(k^a)^a = k^{a^2} = k$.

$\square$

5 Problem 5.5.8

Construct a non-abelian group of order 75. Classify all groups of order 75.

Proof. We have $n_3 = 1, 5, 25$, but 5 is not 1 mod 3, so $n_3 = 1$ or 25. $n_5 = 1, 3$, but 3 is not 1 mod 5, so $n_5 = 1$. In particular, a group of order 75 must have a normal 5-Sylow subgroup. The possible groups of order 25 are $\mathbb{Z}/2\mathbb{Z}5$ and $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$. In the case that $n_3 = 1$, the groups are $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}5$ and $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

If $n_3 = 25$, we must consider semidirect products. Since $|\mathrm{Aut}(\mathbb{Z}/2\mathbb{Z}5)| = \phi(25) = 20$, with ϕ the totient function, there is no order 3 automorphism of $\mathbb{Z}/2\mathbb{Z}5$. Therefore, there is no semidirect product which gives a group that is not isomorphic to one of the previously listed groups.

Let $H = \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$. Since $\mathrm{Aut}(H) \cong GL_2(\mathbb{Z}/5\mathbb{Z})$, $|\mathrm{Aut}(H)| = (5-1)^2(5)(5+1) = 480$, so there is an order 3 automorphism. There are 20 (checked via computer) order 3 automorphisms. The order of 3 in 480 is 1, so the 3-Sylow subgroups of $\mathrm{Aut}(H)$ have order 3. This means that regardless of the choice of $\varphi(1)$ for a nontrivial homomorphism $\varphi : \mathbb{Z}/3\mathbb{Z} \to \mathrm{Aut}(H)$, the image will always be conjugate to the image for any other choice. Therefore, by Problem 5.5.6, there is exactly one $\mathbb{Z}/3\mathbb{Z} \rtimes_\varphi (\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z})$ up to isomorphism, where φ sends a generator of $\mathbb{Z}/3\mathbb{Z}$ to an order 3 automorphism. $\square$