MATH 7210 Homework 5

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1 Problem 1

Let G be a group and let $\Delta = \{(g,g) \mid g \in G\}$.

a) Show that Δ is a subgroup of $G \times G$ which is isomorphic to G.

Proof. Let $f : G \to G \times G$ be $f(g) = (g,g)$. Then $f(gh) = (gh, gh) = (g,g)(h,h) = f(g)f(h)$, so f is a homomorphism. Since $f(G) = \Delta$, it follows Δ is a subgroup of $G \times G$. If $f(g) = (e,e)$, then clearly $g = e$, showing that f is injective. Thus $f : G \to \Delta$ is an isomorphism. □

b) Show that Δ is a normal subgroup iff G is abelian.

Proof. We have $(h_1^{-1}, h_2^{-1})(g,g)(h_1,h_2) = (h_1^{-1}gh_1, h_2^{-1}gh_2)$. Then $h_1^{-1}gh_1 = h_2^{-1}gh_2$ iff $h_2h_1^{-1}g = gh_2h_1^{-1}$. Since h_1, h_2 are arbitrary, $h = h_2h_1^{-1}$ is also arbitrary. Thus Δ is normal iff $hg = gh$ for all $g, h \in G$, i.e. G is abelian. □

2 Problem 2

a) Suppose that $N_1, ..., N_k$ are normal subgroups of G such that $G = N_1...N_k$ and $N_i \cap (N_1...N_{i-1}N_{i+1}...N_k) = \{e\}$ for each i. Show that $G \cong N_1 \times ... \times N_k$.

Proof. Let $f : N_1 \times ... \times N_k \to G$ be $f(n_1, ..., n_k) = n_1...n_k$. Since $G = N_1...N_k$, this map is surjective. Thus suppose $n_1...n_k = e$. For each i, we have $n_i^{-1} = n_1...n_{i-1}n_{i+1}...n_k$. The left hand side belongs to N_i, and the right hand side belongs to $N_1...N_{i-1}N_{i+1}...N_k$. Since $N_i \cap (N_1...N_{i-1}N_{i+1}...N_k) = \{e\}$, we must have $n_i^{-1} = e$, so each $n_i = e$. Thus f is injective, and so f is an isomorphism. □

b) Give an example of a group G and normal subgroups N_1, N_2, N_3 such that $G = N_1N_2N_3$ and $N_i \cap N_j = \{e\}$ for $i \neq j$, but G is not isomorphic to $N_1 \times N_2 \times N_3$.

Proof. Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{(0,0), (1,0), (0,1), (1,1)\}$. Then there are three subgroups $N_1 = \{(0,0), (1,0)\}, N_2 = \{(0,0), (0,1)\}, N_3 = \{(0,0), (1,1)\}$ which are pairwise disjoint. Since G is abelian, each of these subgroups is normal. $G = N_1N_2N_3$ since we have $(0,0) = (0,0) + (0,0) + (0,0), (1,0) = (1,0) + (0,0) + (0,0), (0,1) = (0,0) + (0,1) + (0,0), (1,1) = (0,0) + (0,0) + (1,1)$. However, $N_1 \times N_2 \times N_3$ is a group of order 8, so it is not isomorphic to G. □

3 Problem 3

Let A be abelian and let $\hat{A} = \mathrm{Hom}(A, \mathbb{C}^*)$.

a) Show that $\hat{A}$ is a group under pointwise multiplication.

Proof. Let $\alpha, \beta \in \hat{A}$. First we must show that $(\alpha\beta)(a) = \alpha(a)\beta(a)$ is an element of $\hat{A}$. We have $(\alpha\beta)(ab) = \alpha(ab)\beta(ab) = \alpha(a)\alpha(b)\beta(a)\beta(b) = \alpha(a)\beta(a)\alpha(b)\beta(b) = (\alpha\beta)(a)(\alpha\beta)(b)$, as desired. Clearly, this multiplication is associative since multiplication in $\mathbb{C}^*$ is associative. We also note that the multiplication in $\hat{A}$ is commutative, since multiplication in $\mathbb{C}^*$ is commutative.

The identity in $\hat{A}$ is $\varepsilon(a) = 1$. This is clearly an element of $\hat{A}$ since $\varepsilon(ab) = 1 = \varepsilon(a)\varepsilon(b)$. Furthermore, it is identity since $(\varepsilon\alpha)(a) = 1 \cdot \alpha(a) = \alpha(a)$.

Let α^{-1} be $\alpha^{-1}(a) = (\alpha(a))^{-1}$. This is in $\hat{A}$ since $\alpha^{-1}(ab) = (\alpha(ab))^{-1} = (\alpha(a)\alpha(b))^{-1} = (\alpha(a))^{-1}(\alpha(b))^{-1} = \alpha^{-1}(a)\alpha^{-1}(b)$. This is the inverse of α because $\alpha(a)\alpha^{-1}(a) = \alpha(a)(\alpha(a))^{-1} = 1 = \varepsilon(a)$. □

b) If A is cyclic of order n, show that $\hat{A}$ is also cyclic of order n.

Proof. Let $A = \{e, a, ...a^{n-1}\}$. Then given $\alpha \in \hat{A}$, $\alpha(a^k) = (\alpha(a))^k$, so α is completely determined by $\alpha(a)$. Furthermore, $(\alpha(a))^n = \alpha(e) = 1$, so $\alpha(a)$ is an nth root of unity. Each of the n nth roots of unity determine a unique element of $\hat{A}$, showing that $\hat{A}$ is cyclic of order n, having generator $\alpha(a) = \exp(2\pi i/n)$. □

c) Show that $\widehat{A \times B} \cong \hat{A} \times \hat{B}$.

Proof. Let $f : \hat{A} \times \hat{B} \to \widehat{A \times B}$ be $f(\alpha, \beta)(a, b) = \alpha(a)\beta(b)$. Then $f(\alpha\gamma, \beta\delta)(a, b) = (\alpha\gamma)(a)(\beta\delta)(b) = \alpha(a)\gamma(a)\beta(b)\delta(b) = \alpha(a)\beta(b)\gamma(a)\delta(b) = f(\alpha, \beta)(a, b)f(\gamma, \delta)(a, b)$, showing that f is a homomorphism.

If $f(\alpha, \beta)(a, b) = 1$ for all $(a, b) \in A \times B$, then we consider (a, e) and (e, b), showing that $\alpha(a) = 1$ and $\beta(b) = 1$ for all $a \in A$ and $b \in B$. Thus α and β are identity in $\hat{A}$ and $\hat{B}$ respectively, showing that f is injective.

Let $g \in \widehat{A \times B}$. Then $g_A(a) = g(a, e)$ gives an element of $\hat{A}$; similarly, $g_B(b) = g(e, b)$ gives an element of $\hat{B}$. We claim $f(g_A, g_B) = g$. Indeed, $f(g_A, g_B)(a, b) = g_A(a)g_B(b) = g(a, e)g(e, b) = g(ae, eb) = g(a, b)$, as desired. Thus f is surjective, showing that f is an isomorphism. □

d) Using b) and c), show that $A \cong \hat{A}$ for A finite.

Proof. Since finite abelian groups are isomorphic to direct products of cyclic groups, say $A \cong \Pi_n \mathbb{Z}/n\mathbb{Z}$, then $\hat{A} \cong \Pi_n \widehat{\mathbb{Z}/n\mathbb{Z}} \cong \Pi_n \mathbb{Z}/n\mathbb{Z} \cong A$. □

e) Show that $\hat{\mathbb{Z}} \cong \mathbb{C}^*$.

Proof. Note that $\alpha(n) = (\alpha(1))^n$. Thus, an element of $\hat{\mathbb{Z}}$ is completely determined by $\alpha(1) \in \mathbb{C}^*$. Furthermore, there is no restriction on $\alpha(1)$. Thus, nonzero complex number gives rise to an unique element of $\hat{\mathbb{Z}}$, so that $\hat{\mathbb{Z}} \cong \mathbb{C}^*$. □

f) Show that $f : A \to \hat{\hat{A}}$ given by $f(a)(\alpha) = \alpha(a)$ is a homomorphism.

Proof. $f(ab)(\alpha) = \alpha(ab) = \alpha(a)\alpha(b) = f(a)(\alpha)f(b)(\alpha) = (f(a)f(b))(\alpha)$. □

g) If A is finite, show that f is an isomorphism.

Proof. Since $|A| = |\hat{A}| = |\hat{\hat{A}}|$, it suffices to show that f is injective. Suppose $e \neq a \in A$. Let $A = \Pi_n \mathbb{Z}/n\mathbb{Z}$. a has a non-zero projection into some $\mathbb{Z}/k\mathbb{Z}$, say j mod k. $\mathbb{Z}/k\mathbb{Z}$ admits a homomorphism ψ sending 1 mod k to $\exp(2\pi i/k)$, which therefore sends j mod k to $\exp(2j\pi i/k) \neq 1$. Then we can lift ψ to A given by taking the identity on every $\mathbb{Z}/n\mathbb{Z}$ component except for $n = k$. In other words, if a is written as $(a_{n_1}, ...)$, a is mapped to $\psi(a_j) \neq 1$. Thus $f(a)(\psi) \neq 1$, so that $f(a)$ is not the identity in $\hat{\hat{A}}$. Therefore, f is injective. □

4 Problem 4

a) Show that there are no simple groups of order 312.

Proof. $312 = 2^3 \cdot 3 \cdot 13$. $n_{13} \mid 24$ and $n_{13} \equiv 1 \bmod 13$, implying that $n_{13} = 1$. Thus any group of order 312 has a normal subgroup of order 13. □

b) Show that there are no simple groups of order 56.

Proof. $56 = 2^3 \cdot 7$. $n_7 \mid 8$ and $n_7 \equiv 1 \bmod 7$ implies $n_7 = 1$ or 8. If $n_7 = 1$ then the group is not simple. If $n_7 = 8$, then there are 48 elements of order 7, leaving 8 elements. These must constitute the Sylow subgroup of order 8, implying that $n_2 = 1$, so the group is not simple. □

c) How many elements of order 7 does a simple group of order 168 have?

Proof. $168 = 2^3 \cdot 3 \cdot 7$. $n_7 \mid 24$ and $n_7 \equiv 1 \bmod 7$, so $n_7 = 1$ or 8. If $n_7 = 1$, then the group is not simple. Thus $n_7 = 8$. Then there are 48 elements of order 7. □

5 Problem 4.5.16

Let $|G| = pqr$ where p, q, r are primes with $p < q < r$. Prove that G has a normal Sylow subgroup for either p, q, or r.

Proof. First, $n_r \mid pq$ and $n_r \equiv 1 \bmod r$, meaning that $n_r = 1$ or $n_r = pq$ if $pq \equiv 1 \bmod r$. $n_r \neq p, q$ since $p, q < r$ implies $p, q \not\equiv 1 \bmod r$. If $n_r = 1$, then there is a normal Sylow r-subgroup. Thus suppose $n_r = pq$, so there are $pq(r-1)$ elements of order r.

Now $n_q \mid pr$ and $n_q \equiv 1 \bmod q$. $n_q \neq p$ since $p < q$ implies $p \not\equiv 1 \bmod q$. If $n_q = pr$, then there would be $pr(q-1)$ elements of order q. However, since there are $pqr - pq$ elements of order r and pqr elements in total, then we must have $pr(q-1) < pq$, or $r(q-1) < q$. Since $r > q > 1$, this is impossible. If $n_q = 1$, then there is a normal Sylow q-subgroup. Thus suppose $n_q = r$, so there are $r(q-1)$ elements of order q.

We must have at least $p-1$ elements of order p, and the identity of order 1. Thus $pq(r-1)+r(q-1)+(p-1)+1 \leq pqr$. Then $0 \leq pq-p-qr+r = (p-r)(q-1)$. Since $p < r$ and $q > 1$, this is impossible. Therefore, $n_q = 1$ when $n_r = pq$. $\square$