MATH 7210 Homework 4

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1 Problem 3

If $n \geq 3$, is S_n isomorphic to $A_n \times S_2$?

Proof. No. Write $S_2 = \{1, -1\}$. Then $(e, -1) \in Z(A_n \times S_2)$, because $e \in Z(A_n)$ and $-1 \in Z(S_2)$. That is to say, $Z(A_n \times S_2)$ is not trivial, since it contains a non-identity element. However, S_n for $n \geq 3$ has a trivial center: Suppose that $e \neq \sigma \in S_n$. Then there exist $i \neq j$ in $\{1, ..., n\}$ such that $\sigma(i) = j$. Consider $k \in \{1, ..., n\}$ which is neither i nor j; this exists since $n \geq 3$. Then let $\tau = (jk)$. $\sigma\tau(i) = \sigma(i) = j$, while $\tau\sigma(i) = \tau(j) = k$. Thus $\sigma\tau \neq \tau\sigma$, showing that an arbitrary non-identity permutation $\sigma \notin Z(S_n)$. Since isomorphic groups have isomorphic centers, this shows S_n and $A_n \times S_2$ are not isomorphic. $\square$

2 Problem 4

Let G be a group of order $n \equiv 2 \mod 4$ and let $\theta : G \to S_n$ be the left regular permutation representation of G.

(a) If $g \in G$ has order 2, show that $\theta(g)$ is an odd permutation by explicitly finding the cycle decomposition.

Proof. For any $h \in G$, $h \mapsto gh$ and $gh \mapsto g^2h = h$. Thus $\theta(g)$ is a product of $n/2$ disjoint transpositions, corresponding to an element h being switched with gh. Since $n/2$ is odd, $\theta(g)$ is an odd permutation. $\square$

(b) Show that G has a normal subgroup of index 2, so that it is not simple.

Proof. Consider the composition $\Theta = \mathrm{sgn}\theta : G \to \{\pm 1\}$. Then from (a) we have $\Theta(g) = -1$, implying that Θ is surjective. Then $\ker \Theta$ is a normal subgroup of index 2, since $G/\ker\Theta \cong \{\pm 1\}$. $\square$

3 Problem 4.3.19

Assume $H \lhd G$, $\mathcal{K}$ is a conjugacy class of G contained in H and $x \in \mathcal{K}$. Prove that $\mathcal{K}$ is a union of k conjugacy classes of equal size in H, where $k = |G : HC_G(x)|$. Deduce that a conjugacy class in S_n which consists of even permutations is either a single conjugacy class under A_n or is a union of two classes of the same size in A_n.

Proof. Let $g_1 HC_G(x), ..., g_k HC_G(x)$ be the k disjoint cosets of $HC_G(x)$ in G. Let $\mathcal{C}_i$ be the H-conjugacy class of $g_i x g_i^{-1} \in \mathcal{K}$. Clearly $\mathcal{C}_i \subset \mathcal{K}$. First, we must show this is independent of the coset representative g_i. Let $h \in H, c \in C_G(x)$. Then $g_i hcxc^{-1}h^{-1}g_i^{-1} = h' g_i x g_i^{-1} h'^{-1} \in \mathcal{C}_i$, where $h' = g_i h g_i^{-1} \in H$ by normality of H.

Now suppose there is $h \in H$ such that $g_i x g_i^{-1} = h g_j x g_j^{-1} h^{-1}$. Then $g_i^{-1} h g_j \in C_G(x)$, say $g_i^{-1} h g_j = c$. Then $g_i c = g_i e c \in g_i HC_G(x)$ and $hg_j = g_j h' e \in g_j HC_G(x)$, where h' is given by normality of H. Thus $g_i c = hg_j$ is in two cosets of $HG_C(x)$, implying $i = j$. In particular, this shows that $\mathcal{C}_i$ and $\mathcal{C}_j$ for $i \neq j$ are disjoint.

Since each $\mathcal{C}_i \subset \mathcal{K}$, $\bigcup_{i=1}^k \mathcal{C}_i \subset \mathcal{K}$. Now let $gxg^{-1} \in \mathcal{K}$. g is in some coset $g_i HC_G(x)$; let $g = g_i hc$ for $h \in H, c \in C_G(x)$. Let $g_i h = h' g_i$ by normality of H. Then $gxg^{-1} = h' g_i cxc^{-1} g_i^{-1} h'^{-1} = h' g_i x g_i^{-1} h'^{-1} \in \mathcal{C}_i$. Thus every element of $\mathcal{K}$ belongs to some $\mathcal{C}_i$, so $\mathcal{K} = \bigcup_{i=1}^k \mathcal{C}_i$.

In the case of $G = S_n, H = A_n$, we have $k = |G : HC_G(x)|$ is either 1 or 2, since $A_n = H \leq HC_G(x)$, so $k \leq |S_n : A_n| = 2$. $\square$

4 Problem 6

Let $\mathcal{C}$ be a S_n conjugacy class contained in A_n. Let σ be a fixed element in $\mathcal{C}$.

(a) Show that $\mathcal{C}$ is a single A_n conjugacy class iff $C_{S_n}(\sigma)$ contains an odd permutation.

Proof. ($\rightarrow$) Suppose $C_{S_n}(\sigma)$ contains only even permutations, i.e. $C_{S_n}(\sigma) \leq A_n$. Then $A_n C_{S_n}(\sigma) = A_n$, meaning that the index $|S_n : A_n C_{S_n}(\sigma)| = 2$, so that $\mathcal{C}$ consists of 2 A_n conjugacy classes.

($\leftarrow$) If $C_{S_n}(\sigma)$ contains an odd permutation, then $A_n C_{S_n}(\sigma)$ contains both A_n and an odd permutation. Since $A_n C_{S_n}(\sigma)$ is either A_n or S_n, it must be S_n since A_n does not contain odd permutations. Then $|S_n : A_n C_{S_n}(\sigma)| = 1$, so $\mathcal{C}$ is a single conjugacy class in A_n. □

(b) Show that if the cycle type of σ does not consist of distinct odd integers, then $C_{S_n}(\sigma)$ contains an odd permutation.

Proof. First assume the cycle type σ has an even integer. The corresponding cycle is an odd permutation. Since the cycles in the decomposition of σ are disjoint, said even length cycle commutes with σ, so that it is contained in $C_{S_n}(\sigma)$.

Now suppose that the cycle type contains two cycles with the same odd length k, say $(i_1 ... i_k), (j_1 ... j_k)$. Let $\tau = (i_1 j_1)...(i_k j_k)$. Then $\sigma i_r = i_{r+1}, \sigma j_r = j_{r+1}, \tau i_r = j_r, \tau j_r = i_r$, where for notational purposes, $i_{k+1} = i_1, j_{k+1} = j_1$. Note that τ leaves every element other than the i, j's fixed, so we must check that $\sigma\tau = \tau\sigma$ just on the i, j's. Indeed, $\sigma\tau i_r = \sigma j_r = j_{r+1}$ and $\tau\sigma i_r = \tau i_{r+1} = j_{r+1}$, and $\sigma\tau j_r = i_{r+1} = \tau\sigma j_r$. Thus $\tau \in C_{S_n}(\sigma)$. Since τ is a product of an odd number of transpositions, it is an odd permutation, showing that $C_{S_n}(\sigma)$ has an odd permutation. □

(c) If the cycle type of σ consists of distinct odd integers, show that $C_{S_n}(\sigma)$ is the subgroup of A_n generated by the cycles of σ.

Proof. There is a clear containment of the subgroup generated by the cycles, since each cycle commutes with σ. Thus let $\tau \in C_{S_n}(\sigma)$. Suppose the cycles of σ are denoted by superscript j, and let $(i_1^j ... i_r^j)$ be such a cycle. Let $\tau i_k^j = i_{k'}^{j'}$. Then $\tau\sigma i_k^j = \tau i_{k+1}^j$ and $\sigma\tau i_k^j = \sigma i_{k'}^{j'} = i_{k'+1}^{j'}$. Thus $\tau i_{k+1}^j = i_{k'+1}^{j'}$. Again, if necessary, the $+1$ wraps around the end of a cycle; $i_{r+1}^j = i_1^j$, for example. Therefore, we have $\tau i_1^j = i_{1'}^{j'}, ..., \tau i_r^j = i_{r'}^{j'}$, and $i_{r'+1}^{j'} = i_{1'}^{j'}$. In other words, the $i^{j'}$ form a length r cycle under σ. But there is only one such cycle. Therefore, $j' = j$. It follows that each cycle of τ is a power of a cycle in σ, so that τ is generated by the cycles of σ. □

5 Problem 4.6.4

Prove that A_n is generated by the set of all 3-cycles for each $n \geq 3$.

Proof. S_n is generated by transpositions, and in particular, the transpositions $(12), ..., (1n)$ generate S_n since a transposition $(ij) = (1i)(1j)(1i)$ for $i, j \neq 1$, and the cases where i or j equal 1 are trivial: $(i1) = (1i)$, $(11) = \text{id}$. Thus we can write a permutation as $(1i_1)\dots(1i_k)$. If $i_j = i_{j+1}$, then $(1i_j)(1i_{j+1}) = \text{id}$, so we can assume that consecutive i's are distinct. Then a permutation in A_n is expressed as $(1i_1)...(1i_{2k})$. There are then k pairs $(1i_j)(1i_{j+1}) = (1i_{j+1}i_j)$, showing that any permutation in A_n is a product of three-cycles as desired. $\square$