MATH 7210 Homework 11

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1 Problem 1

Let R be a PID, and M a finitely generated torsion R-module. Let n be the number of invariant factors of M, and let m be the number of elementary divisors of M.

a) If $M = M_1 \oplus \cdots \oplus M_s$, where the M_i's are nonzero cyclic modules, show that $n \leq s \leq m$.

Proof. □

b) Show that if $s = m$, then the given decomposition is the elementary divisor decomposition, up to order.

Proof. □

c) Give an example of a finite abelian group and a direct sum decomposition which is not the invariant factor decomposition, up to order, but with $s = n$.

Proof. □

2 Problem 2

Let R be a PID, and M, N two finitely generated torsion R-modules. Show that if $M \oplus M \cong N \oplus N$, then $M \cong N$.

Proof. Let $M = R/(m_1) \oplus \cdots \oplus R/(m_j)$ be the invariant factor decomposition of M; let $N = R/(n_1) \oplus \cdots \oplus R/(n_k)$ be the invariant factor decomposition of N. Then $M \oplus M = R/(m_1) \oplus R/(m_1) \oplus \cdots R/(m_j) \oplus R/(m_j)$ by just rearranging summands. Furthermore, since $m_1 \mid m_2 \mid ... \mid m_j$, we have $m_1 \mid m_1 \mid ... \mid m_j \mid m_j$, so we have $M \oplus M$ in invariant factor decomposition. Similar result holds for $N \oplus N$. Since $M \oplus M \cong N \oplus N$, we have that the list $m_1, m_1, ..., m_j, m_j$ is the same as the list $n_1, n_1, ..., n_k, n_k$, implying that the list $m_1, ..., m_j$ is the same as the list $n_1, ..., n_k$, implying that $M \cong N$. □

3 Problem 3

a) What are the possible minimal polynomials of an idempotent matrix?

Proof. Let $p(X) = X^2 - X$. If A is idempotent, then by definition $p(A) = 0$. However, if A is the identity, which is idempotent, then the minimal polynomial is $X - 1$. If $A = 0$, which is also idempotent, the minimal polynomial is X. Clearly, these are the only two matrices which satisfy these two polynomials, so in all other cases the minimal polynomial is $p(X) = X^2 - X$. □

b) Show that an idempotent matrix is diagonalizable.

Proof. The possible minimal polynomials $X, X - 1, X^2 - X$ all have roots with multiplicity 1, implying that the generalized eigenvalues of A all have one-dimensional generalized eigenspaces. Thus, the Jordan canonical form of A is made up of 1 by 1 blocks; the Jordan form is diagonal. □

c) Show that two idempotent matrices are similar iff they have the same rank.

Proof. Note that the only eigenvalues of an idempotent matrix are $0, 1$. Since the Jordan form is diagonal, it consists of m 1's along the diagonal and 0's everywhere else. The rank of such a matrix is clearly m, which is also the rank of A. Furthermore, the number m of 1's along the diagonal uniquely determines the Jordan form. Thus, if the rank of A is m, the Jordan form must have m 1's. Since two matrices are similar iff they have the same Jordan form, two idempotent matrices are similar iff they have the same rank. □

4 Problem 4

Let $A \in M_{n \times n}(F)$ be the matrix with all entries equal to 1.

a) Determine the Jordan canonical form if $F = \mathbb{Q}$.

Proof. A simple calculation shows that $A^2 = nA$. Then the minimal polynomial is $X^2 - nX$, except when $n = 1$, in which the minimal polynomial is $X - 1$. Note that $A \neq nI, A \neq 0$, so that $X - n$ and X are not the minimal polynomial. In that case, A is obviously already in Jordan form. Thus let $n > 1$. Now we see that A has eigenvalues 0 and n, with geometric multiplicities of 1 and 1 respectively. It follows that the Jordan blocks must all be 1 by 1. A can be row reduced to a matrix with a single nonzero row, so the rank of A is 1. Then the Jordan form also has rank 1, meaning it must be a diagonal matrix with one n and 0's everywhere else. □

b) Determine the Jordan canonical form if $F = \mathbb{Z}/p\mathbb{Z}$, where p is prime.

Proof. First assume p does not divide n. Then everything from part a applies, and the Jordan form is diagonal with one n and 0's everywhere else. If $p \mid n$, then we have $A^2 = 0$, so the minimal polynomial is X^2 (and not X, since $A \neq 0$). Thus all the eigenvalues are 0, and there is at least one 2 by 2 Jordan block. Since the rank of A is 1, there can only be one 2 by 2 Jordan block, since this will give the single non-zero row in the Jordan form of A. Thus, the Jordan form of A consists of a single 1 on the off-diagonal, and 0's everywhere else. □