MATH 4035 Homework 9

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1 Problem 10.4

If $L, L' \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)$, then $||L + L'|| \leq ||L|| + ||L'||$.

Proof. For each $\mathbf{x} \in \mathbb{E}^n$, we have $||L(\mathbf{x}) + L'(\mathbf{x})|| \leq ||L(\mathbf{x})|| + ||L'(\mathbf{x})|| \leq ||L|| \cdot ||\mathbf{x}|| + ||L'|| \cdot ||\mathbf{x}|| = (||L|| + ||L'||)||\mathbf{x}||$, with the first inequality due to the triangle inequality on $\mathbb{E}^m$, and the second due to the definition of the norm of L and L'. Since $||L + L'||$ is the infimum of K such that $||L(\mathbf{x}) + L'(\mathbf{x})|| \leq K||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{E}^n$, it follows that $||L + L'|| \leq ||L|| + ||L'||$. $\square$

2 Problem 10.8

Let $L \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)$ and $T \in \mathcal{L}(\mathbb{E}^m, \mathbb{E}^k)$.

a) Prove that $||T \circ L|| \leq ||T|| \cdot ||L||$.

Proof. $||T \circ L(\mathbf{x})|| \leq ||T|| \cdot ||L(\mathbf{x})|| \leq ||T|| \cdot ||L|| \cdot ||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{E}^n$. Since $||T \circ L||$ is the infimum of all values of K satisfying $||T \circ L(\mathbf{x})|| \leq K||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{E}^n$, it follows that $||T \circ L|| \leq ||T|| \cdot ||L||$. □

b) Let $k = m = n$. Denote $T^2 = T \circ T$ and $T^{j+1} = T^j \circ T$. Show that $||T^j|| \leq ||T||^j$.

Proof. From above, with $T = L$, we have $||T^2|| \leq ||T||^2$. Then, by induction, we have $||T^{j+1}|| = ||T^j \circ T|| \leq ||T^j|| \cdot ||T|| \leq ||T||^j \cdot ||T|| = ||T||^{j+1}$. □

c) Give an example in which $||T^2|| < ||T||^2$.

Proof. Consider the linear transformation $T : (x, y) \mapsto (y, 0)$. Then $T^2 : (x, y) \mapsto (y, 0) \mapsto (0, 0)$, so $T^2 = 0 \in \mathcal{L}(\mathbb{E}^2)$. However, $||T|| = 1$, so $||T^2|| = 0 < 1 = ||T||^2$. □

3 Problem 10.16

Prove that the determinant function $\det : \mathcal{L}(\mathbb{E}^n) \to \mathbb{R}$ is continuous.

Proof. Since the determinant of a linear transformation is a polynomial of the coefficients of the matrix of the transformation, which are the coordinates for $\mathcal{L}(\mathbb{E}^n)$, and polynomials are continuous functions, it follows that the determinant is continuous. □

4 Problem 10.20

Prove that $\mathcal{GL}(n, \mathbb{R})$ is an open subset of $\mathcal{L}(\mathbb{E}^n)$.

Proof. We know that $\mathcal{GL}(n, \mathbb{R})$ consists precisely of linear transformations with non-zero determinant. That is, $\mathcal{GL}(n, \mathbb{R}) = \det^{-1}((-\infty, 0) \cup (0, \infty))$. As proved in problem 10.16, the determinant is continuous. Furthermore, $(-\infty, 0) \cup (0, \infty)$ is open. Thus $\mathcal{GL}(n, \mathbb{R}) = \det^{-1}((-\infty, 0) \cup (0, \infty))$ is open as claimed, since for a continuous function, the inverse image of an open set is open. $\square$