MATH 4035 Homework 8

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1 Problem 9.43

a) Prove that every convex set is connected.

Proof. Suppose D is a convex set which is disconnected. Let $D = E_1 \dot{\cup} E_2$ be a separation of D. Let $\mathbf{a} \in E_1, \mathbf{b} \in E_2$. Let $\phi(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$. For each $t \in [0,1]$, $\phi(t)$ is either in E_1 or E_2. ϕ is a continuous function, and $[0,1]$ is connected, so $\phi([0,1])$ is connected as well. Therefore $\phi([0,1])$ cannot be separated, so that $\phi([0,1])$ would have to be contained in either E_1 or E_2. However, $\phi(0) \in E_1, \phi(1) \in E_2$, which contradicts this. Thus D cannot exist; there are no convex sets which are disconnected. □

b) Prove that $\mathbb{E}^n$ is connected for each $n \in \mathbb{N}$.

Proof. Since $\mathbb{E}^n$ is a vector space, $\mathbf{a} + t(\mathbf{b} - \mathbf{a}) \in \mathbb{E}^n$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{E}^n$ and $t \in [0,1]$. Thus $\mathbb{E}^n$ is convex, and by the preceding part, connected. □

2 Problem 9.49

Suppose $D \subset \mathbb{E}^n$ is compact and $\mathbf{f} \in \mathcal{C}(D, \mathbb{E}^m)$ is one-to-one. Prove that if $\mathbf{f}(D)$ is connected, then D is connected.

Proof. By theorem 9.3.3, $\mathbf{f}^{-1}$ is continuous. Since $\mathbf{f}$ is one-to-one, $\mathbf{f}^{-1}(\mathbf{f}(D)) = D$, and therefore by theorem 9.4.1, D is connected. □

3 Problem 9.50

Use theorem 9.4.1 to prove theorem 8.4.2, which states that the graph G_f of a continuous function f is connected.

Proof. Let $\mathbf{g}(x) = (x, f(x))$, where $x \in I$ for a given interval I. Since x and $f(x)$ are continuous functions of x, it follows by part four of theorem 9.2.2 that $\mathbf{g}$ is also continuous. Since I is connected, $g(I)$, which is by construction G_f, is also connected. □