MATH 4035 Homework 7

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1 Problem 9.33

Let $D = \{\frac{1}{n} | n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{E}^1$. Prove the following:

a) D is compact.

Proof. For any $\frac{1}{n}$, we have $\frac{1}{n} > 0 \geq -1$ since $n > 0$, and $\frac{1}{n} \leq 1$ since $n \geq 1$. Also, $-1 \leq 0 \leq 1$. Thus $D \subset [-1, 1]$, so D is bounded.

Notice that 0 is a limit point of D, since for any $\varepsilon > 0$, we can choose $N > \frac{1}{\varepsilon}$ by the Archimedean principle, and then $\frac{1}{N} < \varepsilon$. Thus, there are always non-zero points of D close to 0.

Now consider a non-zero limit point x of D. That is, for any $\varepsilon > 0$, there is some element $d \in D \cap B_\varepsilon(x)$. If $x < 0$, then we can choose $\varepsilon < |x|$, so that $D \cap B_\varepsilon(x) = \emptyset$, since the elements of D are non-negative. Thus $x > 0$. We can similarly choose $\varepsilon \in (0, x)$, so that $0 \notin B_\varepsilon(x)$. Then the only elements in $D \cap B_\varepsilon(x)$ are of the form $\frac{1}{n}$. Since $\varepsilon < x$, the left endpoint of $B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$ is non-zero. Thus, $B_\varepsilon(x)$ contains finitely many terms of D; it contains at most $\left\lfloor \frac{1}{x-\varepsilon} \right\rfloor$ terms by the Archimedean principle. Therefore we can choose the element $\frac{1}{k}$ which has the smallest distance to x. If this distance is non-zero, then we can choose an ε smaller than it, leading to a contradiction of x being a limit point. Therefore, the distance is zero, so $x \in D$.

We have now shown that D contains all of its limit points. Thus D is closed. Thus D is closed and bounded, and therefore compact by the Heine-Borel theorem. □

b) If $\mathbf{f} \in \mathcal{C}(D, \mathbb{E}^m)$, then $||\mathbf{f}||$ must achieve both a maximum and a minimum value on D.

Proof. By problem 9.13, the norm is a continuous function. By problem 9.28, the composition of continuous functions is continuous. Therefore, $||\mathbf{f}|| \in \mathcal{C}(D, \mathbb{E}^1)$. Then since D is compact, the extreme value theorem gives the desired result. □

c) If $\mathbf{g} : D \to \mathbb{E}^m$, then $\mathbf{g} \in \mathcal{C}(D, \mathbb{E}^m)$ if and only if $\mathbf{g}(\frac{1}{n}) \to \mathbf{g}(0)$ as $n \to \infty$.

Proof. Suppose $\mathbf{g} \in \mathcal{C}(D, \mathbb{E}^m)$. 0 is a limit point of D. By theorem 9.2.1, $\lim_{x\to 0} \mathbf{g}(x) = \mathbf{g}(0)$. $(\frac{1}{n})$ is a sequence in $D \backslash \{0\}$ converging to 0. Then by theorem 9.1.1, $\mathbf{g}(\frac{1}{n}) \to \mathbf{g}(0)$.

Now suppose $\mathbf{g}(\frac{1}{n}) \to \mathbf{g}(0)$. Let x_j be any sequence in $D\backslash\{0\}$ converging to 0. Since $D\backslash\{0\} = \{\frac{1}{n}|n \in \mathbb{N}\}$, we may let $x_j = \frac{1}{n_j}$. Let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ such that for all $n > N$, $||\mathbf{g}(\frac{1}{n}) - \mathbf{g}(0)|| < \varepsilon$. Similarly, there is an $M \in \mathbb{N}$ such that for all $j > M$, $\frac{1}{n_j} < \frac{1}{N}$. Then $n_j > N$ for all $j > M$. Then $||\mathbf{g}(x_j) - \mathbf{g}(0)|| < \varepsilon$ for all $j > M$, so $\mathbf{g}(x_j) \to \mathbf{g}(0)$. Since 0 is a limit point of D, we have by theorems 9.2.1 and 9.1.1 that $\mathbf{g} \in \mathcal{C}(D, \mathbb{E}^m)$. □

2 Problem 9.39

Let $D \subseteq \mathbb{E}^n$. Suppose $\mathbf{f}_j \in \mathcal{C}(D, \mathbb{E}^m)$ for all j, and suppose also that $\mathbf{f}_j \to \mathbf{f}$ uniformly on D. Then $\mathbf{f} \in \mathcal{C}(D, \mathbb{E}^m)$.

Proof. Let $\mathbf{x} \in D$ and let $\varepsilon > 0$. There is $N \in \mathbb{N}$ so that for all $n > N$, $||\mathbf{f}_n - \mathbf{f}||_{sup} < \frac{\varepsilon}{3}$. Since $\mathbf{f}_n$ is continuous, there is also a $\delta > 0$ such that for $\mathbf{x}' \in B_\varepsilon(\mathbf{x})$, $||\mathbf{f}_n(\mathbf{x}') - \mathbf{f}_n(\mathbf{x})|| < \frac{\varepsilon}{3}$. Thus

$$||\mathbf{f}(\mathbf{x}') - \mathbf{f}(\mathbf{x})|| \leq ||\mathbf{f}(\mathbf{x}') - \mathbf{f}_n(\mathbf{x}')|| + ||\mathbf{f}_n(\mathbf{x}') - \mathbf{f}_n(\mathbf{x})|| + ||\mathbf{f}_n(\mathbf{x}) - \mathbf{f}(\mathbf{x})|| < \varepsilon,$$

so $\mathbf{f}$ is continuous. □

3 Problem 9.41

Let $D \subseteq \mathbb{E}^n$ and $\mathbf{f} : D \to \mathbb{E}^m$. Prove that if D is compact and $\mathbf{f} \in \mathcal{C}(D, \mathbb{E}^m)$, then $\mathbf{f}$ is uniformly continuous on D.

Proof. Suppose that $\mathbf{f}$ is not uniformly continuous. Then there is an $\varepsilon > 0$ such that for all $\delta > 0$, we can choose $\mathbf{x}, \mathbf{x}' \in D$ with $||\mathbf{x} - \mathbf{x}'|| < \delta$, but $||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}')|| \geq \varepsilon$. Therefore consider a sequence $\delta_n > 0$ which converges to 0, and a sequence of points $\mathbf{x}_n, \mathbf{y}_n \in D$ such that $||\mathbf{x}_n - \mathbf{y}_n|| < \delta_n$ with $||\mathbf{f}(\mathbf{x}_n) - \mathbf{f}(\mathbf{y}_n)|| \geq \varepsilon$ for all n. D is compact, so D is closed and bounded by the Heine-Borel theorem. Since D is bounded, then by the Bolzano-Weierstrass theorem, there is a convergent subsequence $\mathbf{x}_{n_k} \to \mathbf{p}$. Since D is closed, $\mathbf{p} \in D$. Note also that $||\mathbf{y}_{n_k} - p|| \leq ||\mathbf{y}_{n_k} - \mathbf{x}_{n_k}|| + ||\mathbf{x}_{n_k} - \mathbf{p}|| < \delta_n + ||\mathbf{x}_{n_k} - \mathbf{p}|| \to 0 + 0 = 0$, so $\mathbf{y}_{n_k} \to \mathbf{p}$ as well. Since $\mathbf{f}$ is continuous, $\mathbf{f}(\mathbf{p}) = \lim_{k\to\infty} \mathbf{f}(\mathbf{x}_{n_k}) = \lim_{k\to\infty} \mathbf{f}(\mathbf{y}_{n_k})$. Then $||\mathbf{f}(\mathbf{x}_{n_k}) - \mathbf{f}(\mathbf{y}_{n_k})|| \to 0$, which contradicts $||\mathbf{f}(\mathbf{x}_n) - \mathbf{f}(\mathbf{y}_n)|| \geq \varepsilon$ for all n. $\square$