MATH 4035 Homework 6

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1 Problem 9.13

If V is any normed vector space, define $N : V \to \mathbb{R}$ by $N(\mathbf{x}) = ||\mathbf{x}||$. Prove that N is continuous.

Proof. Let $\mathbf{y} \in V$. Let $\varepsilon > 0$. By the triangle inequality, $||\mathbf{x}|| = ||\mathbf{x} - \mathbf{y} + \mathbf{y}|| \leq ||\mathbf{x} - \mathbf{y}|| + ||\mathbf{y}||$. Similarly $||\mathbf{y}|| \leq ||\mathbf{x} - \mathbf{y}|| + ||\mathbf{x}||$. Thus both $||\mathbf{x}|| - ||\mathbf{y}||$ and $||\mathbf{y}|| - ||\mathbf{x}||$, the larger of which is $|\ ||\mathbf{x}|| - ||\mathbf{y}||\ |$, are less than or equal to $||\mathbf{x} - \mathbf{y}||$. Thus for $||\mathbf{x} - \mathbf{y}|| < \varepsilon$, $|N(\mathbf{x}) - N(\mathbf{y})| = |\ ||\mathbf{x}|| - ||\mathbf{y}||\ | \leq ||\mathbf{x} - \mathbf{y}|| < \varepsilon$. Thus N is continuous. □

2 Problem 9.16

Suppose $f : \mathbb{E}^2 \to \mathbb{R}$ has the property that for each fixed value $x_2 = b$ the function $g_b(x_1) = f(x_1, b)$ is a continuous function of x_1. Suppose also that for each fixed value $x_1 = a$ the function $h_a(x_2) = f(a, x_2)$ is a continuous function of x_2. Does it follow that $f(x_1, x_2)$ is continuous?

Proof. It does not follow. Take for example $f(x_1, x_2) = \dfrac{x_1 x_2}{x_1^2 + x_2^2}$ for $(x_1, x_2) \neq (0, 0)$, and $f(0, 0) = 0$. For $b \neq 0$, we have $g_b(x_1) = \dfrac{b x_1}{x_1^2 + b^2}$, which is continuous since $x_1^2 + b^2 \neq 0$ for all $x_1 \in \mathbb{R}$. For $b = 0$, $g_b(x_1) = 0$, which is also continuous. Similarly, for $a \neq 0$, $h_a(x_2) = \dfrac{a x_2}{a^2 + x_2^2}$ is continuous, and for $a = 0$, $h_a(x_2) = 0$ is continuous. However, along the line $x_1 = x_2$, we have $f(x_1, x_2) = \dfrac{x_1^2}{x_1^2 + x_1^2} = \dfrac{1}{2}$, so that along this line $f(x_1, x_2)$ tends to $\frac{1}{2}$ at $(0, 0)$, which is not equal to $f(0, 0)$. Therefore f is not continuous, even though each of the axis sections are. □

3 Problem 9.28

Let $\mathbf{f} : D \to \mathbb{E}^m$ be continuous on its domain $D \subseteq \mathbb{E}^n$, and suppose $\mathbf{g} : \mathbf{f}(D) \to \mathbb{E}^p$ is continuous on $\mathbf{f}(D)$. Prove that $\mathbf{g} \circ \mathbf{f}$ is continuous on D.

Proof. Let $U \subseteq \mathbb{E}^p$ be open. Then $(\mathbf{g} \circ \mathbf{f})^{-1}(U) = \mathbf{f}^{-1}(\mathbf{g}^{-1}(U))$. Since $\mathbf{g}$ is continuous, $\mathbf{g}^{-1}(U)$ is an open subset of $\mathbb{E}^m$ by Theorem 9.2.4. Since $\mathbf{f}$ is also continuous, $\mathbf{f}^{-1}(\mathbf{g}^{-1}(U))$ is an open subset of $\mathbb{E}^n$, again by Theorem 9.2.4. Thus $\mathbf{g} \circ \mathbf{f}$ is continuous by Theorem 9.2.4. □