MATH 4035 Homework 5

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1 Problem 9.4

Let $f : \mathbb{E}^2 \to \mathbb{R}$ by the formula $f(\mathbf{x}) = \dfrac{x_1^2 x_2}{x_1^4 + x_2^2}$ if $\mathbf{x} \neq \mathbf{0}$, and $f(\mathbf{0}) = 0$.

a) Show that if $\mathbf{x} \to \mathbf{0}$ along either the x_1- or x_2-coordinate axis, $f(\mathbf{x}) \to 0$.

Proof. Along the x_1-coordinate axis, we have $\mathbf{x} = (x_1, 0)$, so that $f(\mathbf{x}) = \dfrac{x_1^2 \cdot 0}{x_1^4 + 0^2} = 0$. Thus $f(\mathbf{x}) \to 0$ as $x_1 \to 0$.

Along the x_2-coordinate axis, we have $\mathbf{x} = (0, x_2)$, so that $f(\mathbf{x}) = \dfrac{0^2 \cdot x_2}{0^4 + x_2^2} = 0$. Thus $f(\mathbf{x}) \to 0$ as $x_2 \to 0$. $\square$

b) Show that if $\mathbf{x} \to \mathbf{0}$ along any straight line $x_2 = kx_1$ through the origin, $f(\mathbf{x}) \to 0$.

Proof. Let $\mathbf{x} = (x_1, kx_1)$ with $k \neq 0$, so that $f(\mathbf{x}) = \dfrac{x_1^2 \cdot kx_1}{x_1^4 + k^2 x_1^2} = \dfrac{kx_1}{x_1^2 + k^2}$. Then as $x_1 \to 0$, $f(\mathbf{x}) \to \dfrac{k \cdot 0}{0^2 + k^2} = 0$. $\square$

c) Show that $\lim_{\mathbf{x} \to \mathbf{0}} f(\mathbf{x})$ does not exist.

Proof. Let $\mathbf{x} = (x_1, x_1^2)$. Then $f(\mathbf{x}) = \dfrac{x_1^2 x_1^2}{x_1^4 + x_1^4} = \dfrac{1}{2}$. Thus as $x_1 \to 0$, $f(\mathbf{x}) \to \dfrac{1}{2}$. $\square$

d) Does $\lim_{\mathbf{x} \to \infty} f(\mathbf{x})$ exist? Prove your conclusion.

Proof. $\lim_{\mathbf{x} \to \infty} f(\mathbf{x})$ does not exist. By the above parts, we see that if $\mathbf{x} \to \infty$ on either of the coordinate axes, then $f(\mathbf{x}) \to 0$. However, on the parabola $x_2 = x_1^2$, $f(\mathbf{x}) \to \dfrac{1}{2}$. $\square$

2 Problem 9.5

Let $f : \mathbb{E}^2 \to \mathbb{R}$ by the formula $f(\mathbf{x}) = \dfrac{x_1 x_2^2}{x_1^4 + x_2^2}$ if $\mathbf{x} \neq \mathbf{0}$ and $f(\mathbf{0}) = 0$. Prove that $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x}) = 0$.

Proof. Let $\varepsilon > 0$. Let $||\mathbf{x}|| < \varepsilon$, so that in particular $|x_1| \leq ||\mathbf{x}|| < \varepsilon$. Now $x_1^4 + x_2^2 \geq x_2^2$, so $f(x) \leq \dfrac{x_1 x_2^2}{x_2^2} = x_1$. Thus whenever $\mathbf{x} < \varepsilon$, $|f(x)| \leq |x_1| < \varepsilon$. Therefore $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x}) = 0$. $\square$