MATH 4035 Homework 4

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1 Problem 8.45

a) Prove that every connected subset $S \subseteq \mathbb{E}^1$ is an interval.

Proof. Let $a, b \in S$ with $a \leq b$. Let $c \in (a, b)$. Suppose $c \notin S$. Then $S \subseteq (-\infty, c) \cup (c, \infty)$. Since $a \in S \cap (-\infty, c)$ and $b \in S \cap (c, \infty)$, it follows that $(-\infty, c)$ and (c, ∞) separate S, a contradiction. Thus $c \in S$, so that S is an interval. □

b) Prove that every interval I is a connected subset of $\mathbb{E}^1$.

Proof. Suppose I is disconnected with $I \subseteq A \cup B$ for disjoint open sets A, B. Let $a \in A \cap I$. Suppose without loss of generality that there is a point $b \in B \cap I$ with $a < b$. (Otherwise, we could let $a \in B \cap I$ and $b \in A \cap I$.) Thus $[a, b] \subseteq I$. Since $a \in A$ and $b \in B$, we have that A and B separate $[a, b]$. Let $C = [a, b] \cap A$, $D = [a, b] \cap B$. C is bounded above by b, so let $c = \sup C$. Since $a \in C$ and C is bounded above by b, $c \in [a, b]$. Then $c \in C$ or $c \in D$, but not both, since C and D are disjoint.

Suppose $c \in C$. Then $c \in A$, which is open, so we can find an $r > 0$ such that $(c-r, c+r) \subseteq A$. For instance, $c + \frac{r}{2} \in A$. However, $c + \frac{r}{2} > c = \sup C$, so $c + \frac{r}{2} \notin C$. Then $c + \frac{r}{2} \notin [a, b]$. Since $c \in [a, b]$, this means that $b < c + \frac{r}{2}$. But $c \leq b$, so $b \in [c, c + \frac{r}{2}) \subseteq (c - r, c + r) \subseteq A$, a contradiction.

Thus suppose $c \in D$. Then $c \in B$, which is open, so we can find an $r > 0$ such that $(c - r, c + r) \subseteq B$. Then $c - \frac{r}{2} \in B$. $c - \frac{r}{2}$ is less than c, so it cannot be an upper bound for C. That is, there is some $s \in C$ with $c - \frac{r}{2} \leq s$. Then $s \in [c - \frac{r}{2}, c] \subseteq (c - r, c + r) \subseteq B$, which is a contradiction.

Therefore, the assumption that I is disconnected must be false; I is connected. □

2 Problem 8.47

Prove that $S^1 = \{\mathbf{x} \in \mathbb{E}^2 \mid ||\mathbf{x}|| = 1\}$ is a connected set.

Proof. Let f, g be functions defined on the interval $[-1, 1]$, such that $f(x) = \sqrt{1-x^2}$ and $g(x) = -\sqrt{1-x^2}$. For $\mathbf{x} = (x, y) \in G_f$, we have $y = \sqrt{1-x^2}$, so $x^2 + y^2 = ||\mathbf{x}||^2 = 1$. Since the norm is positive definite, this implies $||\mathbf{x}|| = 1$, so $\mathbf{x} \in S^1$. One similarly shows that $\mathbf{x} \in G_g$ implies $\mathbf{x} \in S^1$. Now suppose $\mathbf{x} = (x, y) \in S^1$. Then $||\mathbf{x}|| = 1$ implies that $x^2 + y^2 = 1$, or $y^2 = 1 - x^2$. This equation splits into the two solutions $y = \sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$. Thus $S^1 = G_f \cup G_g$. Since f, g are continuous, Theorem 8.4.2 implies that G_f and G_g are connected. Furthermore, $G_f \cap G_g = \{(-1, 0), (1, 0)\}$, so Theorem 8.4.3 implies that $G_f \cup G_g = S^1$ is connected. □

3 Problem 8.54

Let $f(x) = \sin\frac{\pi}{x}$ if $x > 0$, and $f(0) = 0$. Prove that the graph G_f is a connected subset of $\mathbb{E}^2$.

Proof. Suppose G_f can be separated by disjoint open sets A, B. Suppose without loss of generality that $(0,0) \in A$. Since A is open, there is an open ball $B_r((0,0)) \subseteq A$ for $r > 0$. By the Archimedean property, we can find $n \in \mathbb{N}$ such that $\frac{1}{r} < n$, or $\frac{1}{n} < r$. Thus $(\frac{1}{n}, 0) \in B_r((0,0))$, so $(\frac{1}{n}, 0) \in A$. Since $f(\frac{1}{n}) = \sin n\pi = 0$, $(\frac{1}{n}, 0) \in G_f$. However, f is continuous on $(0,1]$, so by Theorem 8.4.2, $G_f \backslash \{(0,0)\}$ is connected, and thus cannot be separated. Since $(\frac{1}{n}, 0) \in A \cap G_f \backslash \{(0,0)\}$, it follows that $G_f \backslash \{(0,0)\} \subseteq A$. But $(0,0) \in A$, so $G_f \subseteq A$, a contradiction. Thus G_f is connected. □