MATH 4035 Homework 3

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1 Problem 8.39

Prove that if $E \subseteq \mathbb{E}^n$ is compact, then E is closed.

Proof. Let $\mathbf{p} \in E^c$. Let $\mathbf{x} \in E$. Let $d = ||\mathbf{x} - \mathbf{p}||$. Since $\mathbf{p} \notin E$, $\mathbf{p} \neq \mathbf{x}$, and so $d > 0$. By the Archimedean property, we can choose $k \in \mathbb{N}$ such that $\frac{1}{d} < k$. Thus $\frac{1}{k} < d$, which is the negation of $d = ||\mathbf{x} - \mathbf{p}|| \leq \frac{1}{k}$. Thus $\mathbf{x} \notin \overline{B_{\frac{1}{k}}(\mathbf{p})}$. It follows that the collection of open sets $\overline{B_{\frac{1}{k}}(\mathbf{p})}^c$ for $k \in \mathbb{N}$ is an open cover of E. Since E is compact, we can choose a finite subcover, say $\overline{B_{\frac{1}{k_j}}(\mathbf{p})}^c$ for $j = 1, ..., m$, with $k_1 < ... < k_m$. Thus since $\overline{B_{\frac{1}{k_m}}(\mathbf{p})} \subseteq ... \subseteq \overline{B_{\frac{1}{k_1}}(\mathbf{p})}$, we have that $\overline{B_{\frac{1}{k_1}}(\mathbf{p})}^c \subseteq ... \subseteq \overline{B_{\frac{1}{k_m}}(\mathbf{p})}^c$. Thus $E \subseteq \bigcup_{j=1}^m \overline{B_{\frac{1}{k_j}}(\mathbf{p})}^c = \overline{B_{\frac{1}{k_m}}(\mathbf{p})}^c$. The complement of this relationship is $\overline{B_{\frac{1}{k_m}}(\mathbf{p})} \subseteq E^c$, so $B_{\frac{1}{k_m}}(\mathbf{p}) \subseteq E^c$. Thus for any point in E^c, we can find an open ball around it contained in E^c. Thus E^c is open, so E is closed. $\square$

2 Problem 8.40

Let f be a real-valued function defined on a closed finite interval $[a, b]$ in $\mathbb{E}^1$. Define the graph $G_f = \{(x_1, x_2) \in \mathbb{E}^2 \mid x_2 = f(x_1), x_1 \in [a, b]\}$.

a) Prove that if $f \in \mathcal{C}[a, b]$, then G_f is a compact subset of $\mathbb{E}^2$.

Proof. By Theorem 2.4.1, we know that f is bounded on $[a, b]$; $f(x) \in [m, M]$ for all $x \in [a, b]$. It follows that $G_f \in [a, b] \times [m, M]$, so that G_f is bounded. Consider a cluster point $\mathbf{x} = (x, y)$ of G_f, and suppose that $\mathbf{x} \notin G_f$. Then there is a sequence of points $(x_j, f(x_j))$ in G_f converging to $\mathbf{x}$. Thus $x_j \to x$ and $f(x_j) \to y$ by Theorem 8.1.2. However, since f is continuous, $x_j \to x$ implies that $f(x_j) \to f(x)$. Thus $y = f(x)$, so $\mathbf{x} \in G_f$. This is a contradiction, so G_f must contain all of its cluster points, and so it is closed by Theorem 8.2.3. Thus G_f is closed and bounded, and so it is compact by the Heine-Borel theorem. $\square$

b) Let $g(x_1) = \sin \frac{\pi}{x_1}$ if $x_1 \in (0, 1]$, and $g(0) = 0$. Is the graph G_g a compact subset of $\mathbb{E}^2$? Prove your conclusion.

Proof. G_g is not compact. To show this, we will show that G_g is not closed, where the conclusion will follow by the Heine-Borel theorem. Consider the point $\mathbf{x} = (0, 1)$. Take any open ball $B_r(\mathbf{x})$ for $r > 0$. We can choose $n \in \mathbb{N}$ such that $\dfrac{\frac{1}{r} - \frac{1}{2}}{2} < n$ by the Archimedean principle. This simplifies to $\dfrac{1}{2n + \frac{1}{2}} < r$. The point $\left(\dfrac{1}{2n + \frac{1}{2}}, 1\right)$ is distance $\dfrac{1}{2n + \frac{1}{2}} < r$ away from $\mathbf{x} = (0, 1)$, so $\left(\dfrac{1}{2n + \frac{1}{2}}, 1\right) \in B_r(\mathbf{x})$. However, $g\left(\dfrac{1}{2n + \frac{1}{2}}\right) = \sin\left(\pi(2n + \frac{1}{2})\right) = 1$, so $\left(\dfrac{1}{2n + \frac{1}{2}}, 1\right) \in G_g$. Therefore, $B_r(\mathbf{x}) \cap G_g \neq \emptyset$ for any $r > 0$. This means that $(0, 1)$ is a cluster point of G_g, but since $g(0) = 0$, $(0, 1) \notin G_g$. Therefore G_g is not closed by Theorem 8.2.3, and hence is not compact by the Heine-Borel theorem. $\square$

3 Problem 8.41

Let $E_1 \supseteq E_2 \supseteq ... \supseteq E_k \supseteq ...$ be a decreasing nest of nonempty closed subsets of $\mathbb{E}^n$.

a) Give an example to show it is possible for $\bigcap_{k=1}^{\infty} E_k$ to be empty.

Proof. Define $E_k = [k, \infty)^n$; that is, the product of the interval $[k, \infty)$ with itself for each of the coordinates in $\mathbb{E}^n$. Suppose there was an $\mathbf{x} = (x_1, ..., x_n) \in \bigcap_{k=1}^{\infty} E_k$. Then for all $k \in \mathbb{N}$, $x_1 \geq k$. This contradicts the Archimedean property of $\mathbb{R}$, so $\bigcap_{k=1}^{\infty} E_k = \emptyset$. $\square$

b) If E_1 is compact, show that $\bigcap_{k=1}^{\infty} E_k \neq \emptyset$.

Proof. Suppose that $\bigcap_{k=1}^{\infty} E_k = \emptyset$. Then the complement is $\bigcup_{k=1}^{\infty} E_k^c = \mathbb{E}^n \supseteq E_1$, so that the collection of open sets E_k^c forms an open cover of E_1. Then we can find a finite subcover $E_{k_j}^c$, say for $j = 1, ..., m$, with $k_1 < k_2 < ... < k_m$. Thus we have that $E_{k_1}^c \subseteq ... \subseteq E_{k_m}^c$, meaning that $E_1 \subseteq \bigcup_{j=1}^{m} E_{k_j}^c = E_{k_m}^c$. The complement of this relationship is $E_{k_m} \subseteq E_1^c$, which contradicts $E_{k_m} \subseteq E_1$ under the hypothesis that E_{k_m} is nonempty. Therefore $\bigcap_{k=1}^{\infty} E_k \neq \emptyset$. $\square$