MATH 4035 Homework 2

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September 2020

1 Problem 8.20

Let $\mathcal{O}$ denote an arbitrary open subset of $\mathbb{E}^n$. Prove that $\mathcal{O}$ is the union of a family of open balls.

Proof. Let $\mathbf{x} \in \mathcal{O}$. Since $\mathcal{O}$ is open, there exists some open ball $B_{r_\mathbf{x}}(\mathbf{x}) \subseteq \mathcal{O}$. It follows that $\bigcup_{\mathbf{x}\in\mathcal{O}} B_{r_\mathbf{x}}(\mathbf{x}) \subseteq \mathcal{O}$. However, since any $\mathbf{x} \in \mathcal{O}$ is contained in some ball $B_{r_\mathbf{x}}(\mathbf{x})$, it is then contained in the union $\bigcup_{\mathbf{x}\in\mathcal{O}} B_{r_\mathbf{x}}(\mathbf{x})$. Thus $\mathcal{O} \subseteq \bigcup_{\mathbf{x}\in\mathcal{O}} B_{r_\mathbf{x}}(\mathbf{x})$. It follows that $\mathcal{O}$ is the union of open balls $\bigcup_{\mathbf{x}\in\mathcal{O}} B_{r_\mathbf{x}}(\mathbf{x})$. $\square$

2 Problem 8.26

Define the interior

$$S^\circ = \{\mathbf{x} \in S \mid \exists r > 0 \text{ such that } B_r(\mathbf{x}) \subseteq S\}.$$

Prove that $S \subset \mathbb{E}^n$ is an open set if and only if $S = S^\circ$.

Proof. First suppose S is open. By virtue of set definition of S°, we have $S^\circ \subseteq S$. Now, let $\mathbf{x} \in S$. Since S is open, there is an open ball $B_r(\mathbf{x}) \subseteq S$. It follows that $\mathbf{x} \in S^\circ$, so that $S \subseteq S^\circ$. Thus $S = S^\circ$.

Now suppose $S = S^\circ$. Thus if $\mathbf{x} \in S$, then $\mathbf{x} \in S^\circ$. That is, there is an $r > 0$ such that $B_r(\mathbf{x}) \subseteq S$. This implies that S is open. $\square$

3 Problem 8.28

Define $\overline{S}$ to be the intersection of all closed sets that contain S. Prove that $\overline{S}$ is a closed set and that $\overline{S} = S \cup C$, where C is the set of all cluster points of S.

Proof. By Theorem 8.2.4, since $\overline{S}$ is an intersection of closed sets, it must also be closed.

Now we claim that $S \cup C$ is closed, as follows. In particular, suppose towards contradiction that $S \cup C$ is not closed, so that $\mathbb{E}^n - (S \cup C)$ is not open. Then there exists $\mathbf{p} \in \mathbb{E}^n - (S \cup C)$ such that all open balls around $\mathbf{p}$ intersect $S \cup C$. This is the same as saying that $\mathbf{p}$ is a cluster point of $S \cup C$, since we can form a sequence by taking a family of open balls, such as $B_{\frac{1}{n}}(\mathbf{p})$, and choose points in $S \cup C$ such that the sequence will necessarily converge to $\mathbf{p}$. Since $\mathbf{p}$ was chosen outside of $S \cup C$, the elements of the sequence are never equal to $\mathbf{p}$, so $\mathbf{p}$ is indeed a cluster point.

Now, since $\mathbf{p}$ was chosen outside of C, $\mathbf{p}$ cannot be a cluster point of S. This means we can choose an open ball $B_r(\mathbf{p})$ which does not intersect S, but it has to intersect $S \cup C$. Thus $B_r(\mathbf{p})$ intersects C; say $\mathbf{c} \in B_r(\mathbf{p}) \cap C$. Since $B_r(\mathbf{p})$ is open, and $\mathbf{c}$ is a cluster point of S, there is an open ball $B_s(\mathbf{c}) \subseteq B_r(\mathbf{p})$ which contains a point of S. But if $B_s(\mathbf{c})$ contains a point of S, then so must $B_r(\mathbf{p})$, which is a contradiction. Therefore, $S \cup C$ is closed, and so $\overline{S} \subseteq S \cup C$, since $S \cup C$ is a closed set containing S.

Since $\overline{S}$ is an intersection of sets which contain S, $S \subseteq \overline{S}$. Now consider a point $\mathbf{c} \in C$, and consider an arbitrary closed set D which contains S. $\mathbf{c} \in C$ means that there is some sequence in $S \backslash \{\mathbf{c}\}$ which converges to $\mathbf{c}$. Since D contains S, it follows that this sequence is also a sequence of $D \backslash \{\mathbf{c}\}$; that is to say $\mathbf{c}$ is a cluster point of D. Thus by Theorem 8.2.3, $\mathbf{c} \in D$, since D is closed. $\mathbf{c}$ is an arbitrary element of C, so we have that $C \subseteq D$. Thus $S \cup C \subseteq D$. Taking the intersection of this result over all closed sets D containing S gives $S \cup C \subseteq \overline{S}$. Thus, together with the result of the preceding paragraph, we have $\overline{S} = S \cup C$. □