MATH 4035 Homework 14

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1 Problem 10.76

Let $F(\mathbf{x}) = x_1^{\frac{2}{3}} + x_2^{\frac{2}{3}} + x_3^{\frac{2}{3}} - 1$. Suppose that $c = 0$ for $\mathbf{x}_0 = (a, b, c)$. Show that $F(\mathbf{x}) = 0$ cannot be solved uniquely for x_3 as a $\mathcal{C}^1$ function in terms of x_1, x_2, even when the variables are restricted to suitable open neighborhoods of $c \in \mathbb{R}$ and $(a, b) \in \mathbb{E}^2$. At that same point, are the other two variables locally unique functions of the other two?

Proof. Upon rearranging $F(\mathbf{x}) = 0$, we have $x_3^2 = (1 - x_1^{\frac{2}{3}} - x_2^{\frac{2}{3}})^3$. This admits two solutions, $x_3 = (1 - x_1^{\frac{2}{3}} - x_2^{\frac{2}{3}})^{\frac{3}{2}}$ and $x_3 = -(1 - x_1^{\frac{2}{3}} - x_2^{\frac{2}{3}})^{\frac{3}{2}}$, which are clearly non-negative and non-positive respectively. However, any open set around $c = 0$ will contain positive and negative numbers, so neither solution will suffice.

$\square$

2 Problem 10.81

Let $X \in \mathcal{SL}(2, \mathbb{R})$. Show by applying the Implicit Function Theorem that at least one of the four coordinates of X can be expressed as a $\mathcal{C}^1$ function of the other three coordinates restricted to suitable open sets. Then solve explicitly for the coordinate, without using the Implicit Function Theorem.

Proof. The coordinates of X satisfy $f(x_1, x_2, x_3, x_4) = x_1x_4 - x_2x_3 - 1 = 0$. $f \in \mathcal{C}^1$. Now $\dfrac{\partial f}{\partial x_1} = x_4$, $\dfrac{\partial f}{\partial x_2} = -x_3$, $\dfrac{\partial f}{\partial x_3} = -x_2$, $\dfrac{\partial f}{\partial x_4} = x_1$. If $f(\mathbf{x}) = 0$, then not all of the coordinates x_i can be 0; if they were, then $f(\mathbf{x}) = -1$. Therefore, one of the partial derivatives will be non-zero, and so we can apply the Implicit Function Theorem. The solutions are

$$x_1 = \frac{x_2x_3 + 1}{x_4}, x_4 \neq 0,$$

$$x_2 = \frac{x_1x_4 - 1}{x_3}, x_3 \neq 0,$$

$$x_3 = \frac{x_1x_4 - 1}{x_2}, x_2 \neq 0,$$

$$x_4 = \frac{x_2x_3 + 1}{x_1}, x_1 \neq 0.$$

□

3 Problem 10.82 (b,d,e)

b) Prove that $X \in \mathcal{O}(2) \iff \mathbf{f}(\mathbf{x}) = \mathbf{0} \in \mathbb{E}^3$, where $\mathbf{f}(\mathbf{x}) = (x_1x_3 + x_2x_4, x_1^2 + x_2^2 - 1, x_3^2 + x_4^2 - 1)$ for $\mathbf{x} \in \mathbb{E}^4$.

Proof. $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2^2 & x_1x_3 + x_2x_4 \\ x_1x_3 + x_2x_4 & x_3^2 + x_4^2 \end{pmatrix}$. Then

$$\begin{pmatrix} x_1^2 + x_2^2 & x_1x_3 + x_2x_4 \\ x_1x_3 + x_2x_4 & x_3^2 + x_4^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

if and only if $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ by comparing the entries of the matrix. $\square$

d) Use the Implicit Function Theorem to identify three of the four coordinates of $\mathbf{x}$ that can be expressed as $\mathcal{C}^1$ functions of the remaining coordinate, in a neighborhood of the identity element $I \in \mathcal{O}(2)$.

Proof. $\mathbf{f} \in \mathcal{C}^1$. Consider the Jacobian $\frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_3, x_4)} =$

$$\begin{vmatrix} x_3 & x_1 & x_2 \\ 2x_1 & 0 & 0 \\ 0 & 2x_3 & 2x_4 \end{vmatrix} = 4x_1x_2x_3 - 4x_1^2x_4.$$

Evaluating this at $X = I$ gives $-4 \neq 0$. Therefore the variables x_1, x_3, x_4 can be written in terms of x_2. $\square$

e) Use Equation (10.10) to find the derivatives at the identity element of the three chosen dependent variables with respect to the chosen independent one.

Proof. $\left(\frac{d\mathbf{f}}{d\mathbf{y}}\right) = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, when evaluated at the identity. The inverse of this matrix is $\begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$. Then $\frac{dx_1}{dx_2} = -\frac{1}{2}\frac{df_2}{dx_2} = -x_2 = 0$, $\frac{dx_3}{dx_2} = -\frac{df_1}{dx_2} = -x_4 = -1$, and $\frac{dx_4}{dx_2} = -\frac{1}{2}\frac{df_3}{dx_2} = 0$. $\square$