MATH 4035 Homework 13

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1 Problem 10.62

Give an example of a function $\mathbf{f} \in \mathcal{C}^1(\mathbb{E}^2, \mathbb{E}^2)$ that is locally injective at each point $\mathbf{x} \in \mathbb{E}^2$ for which $x_1 \neq 0$ and $\det \mathbf{f}'(\mathbf{x}) \neq 0$ if $x_1 \neq 0$, yet for which $\mathbf{f}$ is not injective on $\{\mathbf{x} \in \mathbb{E}^2 \mid x_1 > 0\}$.

Proof. Consider $\mathbf{f}(\mathbf{x}) = (x_1 \cos x_2, x_1 \sin x_2)$. Then $\det \mathbf{f}'(\mathbf{x}) = \begin{vmatrix} \cos x_2 & -x_1 \sin x_2 \\ \sin x_2 & x_1 \cos x_2 \end{vmatrix} = x_1$. Then on $D = \{\mathbf{x} \in \mathbb{E}^2 \mid x_1 > 0$, $\det \mathbf{f}'(\mathbf{x}) \neq 0$. The partial derivatives are all continuous functions, so $\mathbf{f} \in \mathcal{C}^1$. Then, by the magnification theorem, $\mathbf{f}$ is locally injective. However, $\mathbf{f}(x_1, x_2) = \mathbf{f}(x_1, x_2 + 2\pi)$, so $\mathbf{f}$ is not injective. $\square$

2 Problem 10.64

Let $f : \mathbb{E}^1 \to \mathbb{E}^1$ be $f(x) = 2x + 4x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Show that f is not injective in any open interval $(-r, r)$ for $r > 0$, although f' exists and is bounded on $(-1, 1)$. Which hypothesis of the Magnification Theorem fails?

Proof. Notice that $2x - 4x^2 \leq f(x) \leq 2x + 4x^2$ for $x \neq 0$. Furthermore, these two parabolas are tangent at the origin, with their derivatives both equal to 2. Since $f(x)$ passes through the origin, and it is sandwiched between the two parabolas, it follows that $f'(0) = 2$. For $x \neq 0$, $f'(x) = 2 + 8x \sin \frac{1}{x} - 4 \cos \frac{1}{x}$, which satisfies $|f'(x)| \leq 6 + 8|x|$. Thus, f' exists and is bounded on $(-1, 1)$.

Let $r > 0$. By the Archimedean property, let $n > \frac{1}{2\pi r}$. Then $\frac{1}{2n\pi} \in (-r, r)$ and $f'(\frac{1}{2n\pi}) = 2 + 8\frac{1}{2n\pi} \sin(2n\pi) - 4\cos(2n\pi) = -2$.

Let $0 < \varepsilon < \min(2, \frac{1}{2n\pi})$. By the definition of the derivative, there is a $\delta_1 > 0$ such that for $|h| < \delta_1$, $|\frac{f(\frac{1}{2n\pi}+h) - f(\frac{1}{2n\pi})}{h} + 2| < \varepsilon$. Also, since f is continuous for $x \neq 0$, there is a $\delta_2 > 0$ such that for $|x - \frac{1}{2n\pi}| < \delta_2$, $|f(x) - f(\frac{1}{2n\pi})| < \varepsilon$.

Now $f(\frac{1}{2n\pi}) = \frac{1}{n\pi} + \frac{1}{n^2\pi^2} \sin(2n\pi) = \frac{1}{n\pi} > \frac{1}{2n\pi} > \varepsilon$. Thus for $|x - \frac{1}{2n\pi}| < \delta_2$, $f(x) > f(\frac{1}{2n\pi}) - \varepsilon > 0$.

Let $0 < h < \min(\delta_1, \delta_2, r - \frac{1}{2n\pi})$. Then $f(\frac{1}{2n\pi} + h) < f(\frac{1}{2n\pi}) + h(\varepsilon - 2) < f(\frac{1}{2n\pi})$, since $\varepsilon < 2$. Since $h < r - \frac{1}{2n\pi}$, $\frac{1}{2n\pi} + h \in (-r, r)$. Furthermore, since $h = |\frac{1}{2n\pi} + h - \frac{1}{2n\pi}| < \delta_2$, it follows that $f(\frac{1}{2n\pi} + h) > 0 = f(0)$.

It has been shown that $f(0) < f(\frac{1}{2n\pi} + h) < f(\frac{1}{2n\pi})$. By the intermediate value theorem, there exists some $c \in (0, \frac{1}{2n\pi})$ with $f(c) = f(\frac{1}{2n\pi} + h)$, so f is not injective on $(-r, r)$.

The magnification theorem can not be applied to f at the origin because $f \notin \mathcal{C}^1$. The derivative $f'(x) = 2 + 8x \sin \frac{1}{x} - 4 \cos \frac{1}{x}$ for $x \neq 0$ is not continuous at $x = 0$, since $\cos \frac{1}{x}$ does not behave well at $x = 0$. □