MATH 4035 Homework 12

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1 Problem 10.59

Suppose $D \subset \mathbb{E}^n$ and $S \subset \mathbb{E}^m$ are both open sets. Suppose $\mathbf{f} : D \to S$ is differentiable, one-to-one and onto S, and suppose $\mathbf{f}^{-1}$ is differentiable also. Prove that $n = m$. Conclude that if $m \neq n$, then $\mathbb{E}^n$ and $\mathbb{E}^m$ are not diffeomorphic, meaning that there is no differentiable one-to-one map of $\mathbb{E}^n$ onto $\mathbb{E}^m$ with differentiable inverse.

Proof. Note that $\mathbf{g} = \mathbf{f} \circ \mathbf{f}^{-1}$, where $\mathbf{g}(\mathbf{x}) = \mathbf{x}$. $\mathbf{g}$ is linear and represented by the identity matrix, so that its derivative is also the identity matrix. Thus $I = \mathbf{f}'(\mathbf{f}^{-1}(\mathbf{x}_0))(\mathbf{f}^{-1})'(\mathbf{x}_0)$ by the chain rule. Similarly, since $\mathbf{g} = \mathbf{f}^{-1} \circ \mathbf{f}$, $I = (\mathbf{f}^{-1})'(\mathbf{f}(\mathbf{x}_0))\mathbf{f}'(\mathbf{x}_0)$. Since we have a bijection, the base-point $\mathbf{x}_0$ does not matter; we can generalize the derivatives by two matrices A, B with $AB = I$, $BA = I$. It follows that both A and B have a trivial kernel, since the existence of a non-zero $\mathbf{v}$ such that $A\mathbf{v} = 0$ or $B\mathbf{v} = 0$ implies $AB\mathbf{v} = 0 = v$ or $BA\mathbf{v} = 0 = v$. Therefore they have maximal rank, which is $\min(m, n)$ by the dimensions of A and B. But, by the rank-nullity theorem applied to A and B, we must have $m = \min(m, n)$ and $n = \min(m, n)$. Thus $m = n$. $\square$

2 Problem 10.61

Suppose $f \in \mathcal{C}(\mathbb{E}^1, \mathbb{E}^1)$ and that f is locally injective, meaning that for each $x \in \mathbb{E}^1$ there is a corresponding $r > 0$ such that f restricted to $B_r(x)$ is injective. Prove that f must be injective on $\mathbb{E}^1$.

Proof. Let $x \in \mathbb{E}^1$ and let $r_1 > 0$ such that f is injective on $B_{r_1}(x)$. Let $a, b \in B_{r_1}(x)$ with $a < b$. Then the three values $f(a), f(b), f(\frac{a+b}{2})$ are distinct by injectivity. Suppose $f(b) > f(\frac{a+b}{2})$.

Suppose further that $f(a) > f(\frac{a+b}{2})$. Let $z \in \mathbb{R}$ such that $z > f(\frac{a+b}{2})$, $z < f(a)$, and $z < f(b)$. Then there exists $c \in (a, \frac{a+b}{2})$ such that $f(c) = z$, and $d \in (\frac{a+b}{2}, b)$ such that $f(d) = z$, by the intermediate value theorem. Since c and d belong to disjoint sets, they are not equal, but they are both elements of $B_{r_1}(x)$, so we have contradicted injectivity on this set. Thus $f(a) < f(\frac{a+b}{2}) < f(b)$.

If we had started with $f(b) < f(\frac{a+b}{2})$, a similar argument can be given to show that if $f(a) < f(\frac{a+b}{2})$ also, we can use the intermediate value theorem to contradict injectivity. Thus $f(a) > f(\frac{a+b}{2}) > f(b)$.

In both cases, we have that the function f is strictly monotone on the open ball $B_{r_1}(x)$. There is an $r_2 > 0$ such that f is injective on $B_r(x + r_1)$, and hence also strictly monotone. This ball intersects $B_{r_1}(x)$ in an interval $(x + r_1 - r_2, x + r_1)$. Since f is strictly monotone of some direction on this subinterval of $B_r(x + r_1)$, it must also be strictly monotone of the same direction on all of $B_r(x + r_1)$. We can repeat this process by letting $r_3 > 0$ such that f is injective on $B_{r_3}(x + r_1 + r_2)$, and so on.

If $r_1 + r_2 + r_3 + ...$ diverges, then we will have that f is strictly monotone on $(x - r_1, \infty)$. If $r_1 + r_2 + r_3 + ...$ converges to s, then we can find an $r > 0$ such that f is injective on $B_r(x + s)$. Since the partial sums of $r_1 + r_2 + r_3 + ...$ are strictly increasing (each term is positive), it follows that the limit s is the supremum of the partial sums. Then $B_r(x + s)$ must intersect some ball $B_{r_{n+1}}(x + r_1 + ... + r_n)$, so f is strictly monotone of the same direction on $B_r(x + s)$. Thus, we can continue indefinitely, so that f is strictly monotone on $(x - r_1, \infty)$. Clearly, we may also apply the same reasoning to the left hand endpoints of the intervals, and extend indefinitely to the left. Thus f is strictly monotone on $\mathbb{E}^1$. A strictly monotone function is injective, since $x \neq y$ implies $x > y$ or $x < y$, which implies $f(x) > f(y)$ or $f(x) < f(y)$, which implies $f(x) \neq f(y)$. Thus f is injective.

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