MATH 4035 Homework 11

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1 Problem 10.45

Suppose that $T \in \mathcal{L}(\mathbb{E}^m, \mathbb{E}^p)$ and that $\mathbf{f} : \mathbb{E}^n \to \mathbb{E}^m$ is differentiable.

a) Prove that $T \circ \mathbf{f}$ is differentiable at each $\mathbf{x} \in \mathbb{E}^n$ and that $(T \circ \mathbf{f})'(\mathbf{x}) = T \circ \mathbf{f}'(\mathbf{x})$.

Proof. By exercise 10.30, T is differentiable. Thus by theorem 10.3.1, $T \circ \mathbf{f}$ is differentiable. The derivative of T at all points is T, so $(T \circ \mathbf{f})'(\mathbf{x}) = T \circ \mathbf{f}'(\mathbf{x})$ by theorem 10.3.1. $\square$

b) If $\mathbf{a} \in \mathbb{E}^m$ is a constant vector, prove that $(\mathbf{a} \cdot \mathbf{f})'(\mathbf{x})$ exists and equals $\mathbf{a} \cdot \mathbf{f}'(\mathbf{x})$.

Proof. The mapping $\mathbf{x} \mapsto \mathbf{a} \cdot \mathbf{x}$ is linear and therefore differentiable. By the preceding work, $(\mathbf{a} \cdot \mathbf{f})'(\mathbf{x}) = \mathbf{a} \cdot \mathbf{f}'(\mathbf{x})$. $\square$

2 Problem 10.50

Suppose that $\mathbf{f}'(\mathbf{x})$ exists and that $||\mathbf{f}'(\mathbf{x})||$ is bounded on a convex set $D \subseteq \mathbb{E}^n$, where $\mathbf{f} : D \to \mathbb{E}^m$. Prove that $\mathbf{f}$ is uniformly continuous on D.

Proof. By the hypotheses, we can apply the Mean Value Theorem, so that $||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|| \leq M||\mathbf{x} - \mathbf{y}||$, where $M = \sup_{\mathbf{x}\in D} ||\mathbf{f}'(\mathbf{x})||$. Let $\varepsilon > 0$. Then let $\delta = \varepsilon/M$. Thus when $||\mathbf{x} - \mathbf{y}|| < \delta$, $||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|| < M\delta = \varepsilon$. □

3 Problem 10.52

Suppose $\mathbf{f}'(\mathbf{x})$ exists for all $\mathbf{x}$ in a nonempty open set $D \subseteq \mathbb{E}^n$, where $\mathbf{f} : D \to \mathbb{E}^m$.

a) Suppose D is convex. If $\mathbf{f}'(\mathbf{x}) = 0$, prove that $\mathbf{f}$ is constant.

Proof. Notice that $\sup_{\mathbf{x}\in D} ||\mathbf{f}'(\mathbf{x})|| = 0$. Thus by the Mean Value Theorem, $||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|| \leq 0$ for all $\mathbf{x}, \mathbf{y} \in D$. Since the norm is a non-negative mapping, this implies $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in D$. □

b) Suppose that D is connected, but not necessarily convex. If $\mathbf{f}'(\mathbf{x}) = 0$, prove that $\mathbf{f}$ is constant.

Proof. Let $\mathbf{x} \in D$. D is open, so there is an $r > 0$ such that $B_r(\mathbf{x}) \subseteq D$. $B_r(\mathbf{x})$ is convex, so by the preceding part $\mathbf{f}$ is constant on this ball. Let $\mathbf{c} \in \mathbb{E}^m$. If $\mathbf{f}^{-1}(\mathbf{c})$ is empty, then it is open. Otherwise, let $\mathbf{x} \in \mathbf{f}^{-1}(\mathbf{c})$. Then there is an $r > 0$ such that $\mathbf{f}$ is constant on $B_r(\mathbf{x})$. Since $\mathbf{x} \in B_r(\mathbf{x})$ and $\mathbf{f}(\mathbf{x}) = \mathbf{c}$, $B_r(\mathbf{x}) \subseteq \mathbf{f}^{-1}(\mathbf{c})$. Thus, $\mathbf{f}^{-1}(\mathbf{c})$ is open. D is connected and therefore cannot be expressed as the disjoint union of non-empty open sets. For $\mathbf{c}_1 \neq \mathbf{c}_2$, $\mathbf{f}^{-1}(\mathbf{c}_1)$ is disjoint to $\mathbf{f}^{-1}(\mathbf{c}_2)$. Since these sets are open, it follows that $D = \mathbf{f}^{-1}(\mathbf{c})$ for some $\mathbf{c}$; that is, $\mathbf{f}$ is constant. □