MATH 4035 Homework 10

Andrea Bourque

October 2020

1 Problem 30

Let $T \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)$. Prove that T is differentiable at each $\mathbf{x} \in \mathbb{E}^n$ and that $T' = T$.

Proof. Since T is linear, $\Delta T(\mathbf{x}) = T(\mathbf{x} + \mathbf{h}) - T(\mathbf{x}) = T(\mathbf{x} + \mathbf{h} - \mathbf{x}) = T(\mathbf{h})$. Thus T is its own differential, so that $T'(\mathbf{x}) = T$. □

2 Problem 32

Suppose that $\mathbf{f}$ and $\mathbf{g} : D \to \mathbb{E}^m$ and that $\mathbf{x} \in D$ is a cluster point of $D \subseteq \mathbb{E}^n$. Let $c \in \mathbb{R}$. If both $\mathbf{f}$ and $\mathbf{g}$ are differentiable at $\mathbf{x}$, prove that $(c\mathbf{f} + \mathbf{g})'(\mathbf{x})$ exists and equals $c\mathbf{f}'(\mathbf{x}) + \mathbf{g}'(\mathbf{x})$.

Proof. $\Delta(c\mathbf{f} + \mathbf{g})(\mathbf{x}) = (c\mathbf{f} + \mathbf{g})(\mathbf{x} + \mathbf{h}) - (c\mathbf{f} + \mathbf{g})(\mathbf{x}) = c(\mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x})) + \mathbf{g}(\mathbf{x} + \mathbf{h}) - \mathbf{g}(\mathbf{x}) = cA_f(\mathbf{h}) + A_g(\mathbf{h}) + c\varepsilon_f(\mathbf{h}) + \varepsilon_g(\mathbf{h})$. Since A_f, A_g are linear, $A = cA_f + A_g$ is linear. Now, $\frac{||c\varepsilon_f(\mathbf{h}) + \varepsilon_g(\mathbf{h})||}{||h||} \leq \frac{c||\varepsilon_f(\mathbf{h})|| + ||\varepsilon_g(\mathbf{h})||}{||\mathbf{h}||} = c\frac{||\varepsilon_f(\mathbf{h})||}{||\mathbf{h}||} + \frac{||\varepsilon_g(\mathbf{h})||}{||\mathbf{h}||} \to c \cdot 0 + 0 = 0$. It follows that $\Delta(c\mathbf{f} + \mathbf{g})(\mathbf{x}) = A(\mathbf{h}) + \varepsilon(\mathbf{h})$, where $\varepsilon = c\varepsilon_f + \varepsilon_g$. Therefore, $(c\mathbf{f} + \mathbf{g})'(\mathbf{x}) = A = cA_f + A_g = c\mathbf{f}'(\mathbf{x}) + \mathbf{g}'(\mathbf{x})$. $\square$

3 Problem 38

Define $f : \mathbb{E}^2 \to \mathbb{E}^1$ by $f(\mathbf{x}) = \dfrac{x_1 x_2}{x_1^2 + x_2^2}$ for $\mathbf{x} \neq 0$ and $f(\mathbf{0}) = 0$.

a) Prove that $\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}$ exist at each $\mathbf{x} \in \mathbb{E}^2$.

Proof.

$$\frac{\partial f}{\partial x_1} = \frac{x_2}{x_1^2 + x_2^2} - \frac{2x_1^2 x_2}{(x_1^2 + x_2^2)^2} = \frac{x_2^3 - x_1^2 x_2}{(x_1^2 + x_2^2)^2}.$$

$$\frac{\partial f}{\partial x_2} = \frac{x_1}{x_1^2 + x_2^2} - \frac{2x_1 x_2^2}{(x_1^2 + x_2^2)^2} = \frac{x_1^3 - x_1 x_2^2}{(x_1^2 + x_2^2)^2}.$$

These formulas are not valid for $\mathbf{x} = \mathbf{0}$. If $x_2 = 0$, then $f(x, 0)$ is constant and equal to 0, so that $\dfrac{\partial f}{\partial x_1} = 0$ for all $\mathbf{x} = (x_1, 0)$. Similarly, $f(0, x)$ is constant and equal to 0, so that $\dfrac{\partial f}{\partial x_2} = 0$ for all $\mathbf{x} = (0, x_2)$. □

b) Prove that f is not differentiable at 0.

Proof. Along the x_1 and x_2-axes, $f(\mathbf{x}) = 0$. However, on the line $x_1 = x_2$, excluding the origin, $f(\mathbf{x}) = \dfrac{1}{2}$. Thus f is not continuous at $\mathbf{0}$, so it cannot be differentiable at $\mathbf{0}$ by Theorem 10.2.2. □

4 Problem 39

Define $f : \mathbb{E}^2 \to \mathbb{E}^1$ by $f(\mathbf{x}) = \dfrac{x_1^2 x_2}{x_1^4 + x_2^2}$ for $\mathbf{x} \neq 0$ and $f(\mathbf{0}) = 0$.

a) Prove that $D_{\mathbf{v}} f(\mathbf{x})$ exists for all $\mathbf{x} \in \mathbb{E}^2$ and $\mathbf{v} \in \mathbb{E}^2 \backslash \{\mathbf{0}\}$.

Proof. In the calculations, we can ignore powers of t higher than 1, since dividing by t and limiting to 0 makes higher powers disappear. We have

$$\frac{(x_1 + tv_1)^2(x_2 + tv_2)}{(x_1 + tv_1)^4 + (x_2 + tv_2)^2} - \frac{x_1^2 x_2}{x_1^4 + x_2^2} = \frac{(x_1^2 + 2tx_1v_1)(x_2 + tv_2)}{x_1^4 + 4tx_1^3v_1 + x_2^2 + 2tx_2v_2} - \frac{x_1^2 x_2}{x_1^4 + x_2^2}$$

$$= \frac{x_1^6x_2 + tx_1^6v_2 + 2tx_1^4x_2v_1 + x_1^2x_2^3 + tx_1^2x_2^2v_2 + 2tx_2^3v_1 - (x_1^6x_2 + 4tx_1^5x_2v_1 + x_1^2x_2^3 + 2tx_1^2x_2^2v_2)}{(x_1^4 + 4tx_1^3v_1 + x_2^2 + 2tx_2v_2)(x_1^4 + x_2^2)}$$

$$= \frac{t(x_1^6v_2 + 2x_1^4x_2v_1 - x_1^2x_2v_2 + 2x_2^3v_1 - 4x_1^5x_2v_1)}{(x_1^4 + 4tx_1^3v_1 + x_2^2 + 2tx_2v_2)(x_1^4 + x_2^2)}.$$

Thus dividing by t and taking $t \to 0$ gives

$$D_{\mathbf{v}} f(\mathbf{x}) = \frac{x_1^6v_2 + 2x_1^4x_2v_1 - x_1^2x_2v_2 + 2x_2^3v_1 - 4x_1^5x_2v_1}{(x_1^4 + x_2^2)^2}.$$

However, this does not account for $\mathbf{x} = \mathbf{0}$. At the origin, we have the difference quotient

$$\frac{1}{t}\frac{(t^2v_1)(tv_2)}{(tv_1)^4 + (tv_2)^2} = \frac{v_1v_2}{t^2v_1^4 + v_2^2} \to \frac{v_1}{v_2}.$$

This exists for $v_2 \neq 0$, which is not a necessary condition. If $v_2 = 0$, then we see that since $f(x, 0)$ is constant and equal to 0, $D_{\mathbf{v}} f(\mathbf{0}) = 0$ for $\mathbf{v} = (v, 0) \neq \mathbf{0}$. □

b) Prove that f is not differentiable at $\mathbf{0}$.

Proof. From exercise 9.4, f is not continuous at $\mathbf{0}$, and therefore cannot be differentiable at $\mathbf{0}$ by Theorem 10.2.2. □