MATH 4035 Homework 1

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1 Problem 8.9

Show that $\mathbf{x}^{(j)} \to \mathbf{x}$ in the sense of the norm of $\mathbb{E}^n$ if and only if the sequence $x_l^{(j)} \to x_l$ for each $l = 1, ..., n$.

Proof. First suppose that $\mathbf{x}^{(j)} \to \mathbf{x}$ in the sense of the norm of $\mathbb{E}^n$. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $m > N$, $||\mathbf{x}^{(m)} - \mathbf{x}|| < \varepsilon$. But $|x_l^{(m)} - x_l| \leq \sqrt{|x_1^{(m)} - x_1|^2 + ... + |x_n^{(m)} - x_n|^2} = ||\mathbf{x}^{(m)} - \mathbf{x}||$, so it follows that each sequence $x_l^{(j)} \to x_l$.

Now suppose that each sequence $x_l^{(j)} \to x_l$. Let $\varepsilon > 0$. For each $l = 1, ..., n$, let $N_l \in \mathbb{N}$ be such that for all $m > N_l$, $|x_l^{(m)} - x_l| < \dfrac{\varepsilon}{\sqrt{n}}$. Take $N = \max(N_1, ..., N_n)$. Thus for all $m > N$, $|x_l^{(m)} - x_l| < \dfrac{\varepsilon}{\sqrt{n}}$ holds for each $l = 1, ..., n$. Thus $||\mathbf{x}^{(m)} - \mathbf{x}|| = \sqrt{|x_1^{(m)} - x_1|^2 + ... + |x_n^{(m)} - x_n|^2} < \sqrt{\dfrac{\varepsilon^2}{n} + ... + \dfrac{\varepsilon^2}{n}} = \varepsilon$. Thus $\mathbf{x}^{(j)} \to \mathbf{x}$. $\square$

2 Problem 8.11

a) Prove that every convergent sequence in $\mathbb{E}^n$ is bounded.

Proof. Suppose $\mathbf{x}^{(j)} \to \mathbf{x}$. Fix some $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $m > N$, $||\mathbf{x}^{(m)} - \mathbf{x}|| < \varepsilon$. Thus $||\mathbf{x}^{(m)}|| \leq ||\mathbf{x}^{(m)} - \mathbf{x}|| + ||\mathbf{x}|| < \varepsilon + ||\mathbf{x}||$. Over all $k < N$, we may take $M = \max(\varepsilon, \max(||\mathbf{x}^{(k)} - \mathbf{x}||))$, so that $||\mathbf{x}^{(j)}|| \leq M + ||\mathbf{x}||$. Thus $\mathbf{x}^{(j)}$ is bounded. $\square$

b) Give an example of a bounded sequence in $\mathbb{E}^2$ that is not convergent.

Proof. We may take $\mathbf{x}^{(j)} = ((-1)^j, -(-1)^j)$. For all j, $||\mathbf{x}^{(j)}|| = \sqrt{2}$, yet there is no limit as the sequence oscillates between $(-1, 1)$ and $(1, -1)$. $\square$

c) Prove that every bounded sequence in $\mathbb{E}^n$ has a convergent subsequence.

Proof. Take $\mathbf{x}^{(j)}$ to be a bounded sequence in $\mathbb{E}^n$; say $||\mathbf{x}^{(j)}|| \leq M$. Then for each $l = 1, ..., n$, $|x_l^{(j)}| \leq ||\mathbf{x}^{(j)}|| \leq M$. Thus $\mathbf{x}^{(j)} \in [-M, M]^n$, a cube of length $2M$. Let $\mathbf{x}^{(n_1)} = \mathbf{x}^{(1)}$. By splitting each interval at the midpoint, we can subdivide this cube into 2^n cubes of length M. One of these cubes must contain ∞-many terms of the sequence; if all of them contained only finitely many terms, then there would be finitely many terms in the sequence. Thus choose any cube which contains a subsequence of $\mathbf{x}^{(j)}$. Since there are ∞-many terms in this cube, we can choose $n_2 > n_1$ such that $\mathbf{x}^{(n_2)}$ is in this smaller cube. We can repeat the process, dividing the M length cube into 2^n cubes of length $\frac{M}{2}$, and choosing one such cube that contains ∞-many terms of the sequence. It follows that if $i, j > k$, then $\mathbf{x}^{(n_i)}$ and $\mathbf{x}^{(n_j)}$ are contained in a cube of length $\frac{2M}{2^k}$, so that $||\mathbf{x}^{(n_i)} - \mathbf{x}^{(n_j)}|| \leq \frac{2M\sqrt{n}}{2^k} \to 0$ as $k \to \infty$. In particular, for any $\varepsilon > 0$, we can choose $K \in \mathbb{N}$ such that for $k > K$, $||\mathbf{x}^{(n_i)} - \mathbf{x}^{(n_j)}|| < \varepsilon$. Thus the subsequence $\mathbf{x}^{(n_i)}$ is Cauchy, and hence converges since $\mathbb{E}^n$ is complete. $\square$

3 Problem 8.14 (a)-(c)

a) Suppose a vector space V is equipped with an inner product $\langle \cdot, \cdot \rangle$, and suppose we define a corresponding norm by $||\mathbf{x}||^2 = \langle \mathbf{x}, \mathbf{x} \rangle$. Prove the Parallelogram Law:

$$||\mathbf{x} + \mathbf{y}||^2 + ||\mathbf{x} - \mathbf{y}||^2 = 2||\mathbf{x}||^2 + 2||\mathbf{y}||^2.$$

Proof. By linearity and symmetry of the inner product, we have $||\mathbf{x} + \mathbf{y}||^2 = \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} + \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle$. Similarly, we have $||\mathbf{x} - \mathbf{y}||^2 = \langle \mathbf{x}, \mathbf{x} \rangle - 2\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle$. Thus upon adding the two equations we obtain the desired result. $\square$

b) Prove that the taxicab norm does not correspond as in part (a) above to any inner product on $\mathbb{R}^2$.

Proof. Suppose that the taxicab norm did correspond to an inner product. Then the result of part (a) would hold true for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. However, choosing, for example, $\mathbf{x} = (1, 0)$ and $\mathbf{y} = (0, 1)$, we have $||\mathbf{x}|| = |1| + |0| = 1 = ||\mathbf{y}||$, $||\mathbf{x} + \mathbf{y}|| = |1| + |1| = 2 = ||\mathbf{x} - \mathbf{y}||$, whence we have $2^2 + 2^2 \neq 2 \cdot 1^2 + 2 \cdot 1^2$. $\square$

c) Under the hypotheses of part (a) above, prove the identity

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4}(||\mathbf{x} + \mathbf{y}||^2 - ||\mathbf{x} - \mathbf{y}||^2).$$

Proof. Using the work under part (a), subtracting the two equations instead of adding gives $||\mathbf{x} + \mathbf{y}||^2 - ||\mathbf{x} - \mathbf{y}||^2 = 4\langle \mathbf{x}, \mathbf{y} \rangle$, which is clearly equivalent to the claim. $\square$